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We study return predictability using a dynamic model of speculative trading among relatively overconfident competitive traders who agree to disagree about the precision of their private information. The return process depends on both parameter values used by traders and empirically correct parameter values. Although traders apply Bayes Law consistently, equilibrium returns are predictable based on current and past dividends and prices. We derive specific conditions under which excess returns exhibit realistic patterns of short-run momentum and long-run mean-reversion. We clarify the concepts of rational expectations and market efficiency in a setting with differences in beliefs.

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It is well known that time series of asset prices exhibit momentum and reversals, but it is usually difficult to construct a satisfactory theoretical explanation for these phenomena. This paper suggests that momentum and, more generally, return predictability may be consequences of the way in which prices aggregate information when traders have heterogeneous beliefs about the accuracy of their private signals. In a setting of perfect competition, the model predicts greater momentum in more liquid markets with larger dispersion in beliefs.

We present a structural model with predictable returns in the equilibrium. Traders in the model mimic the behavior of real-world traders who collect public and private raw information into databases, engage in research to process this information into signals, calculate expected returns or alphas from these signals, and calculate optimal inventories by inputting alphas into risk models. The traders are relatively overconfident. Each trader symmetrically assigns a higher value to the accuracy of his private signal relative to the accuracy of other traders' signals. Since the values traders assign to all economically relevant parameters are common knowledge, as in Aumann (1976), traders agree to disagree about the informativeness of their respective signals. An economist with empirically correct beliefs will typically find returns to be predictable, even when the beliefs of traders in the model are "correct on average." This result contradicts the rational expectations intuition that prices will aggregate fundamental information correctly when traders are correct on average, even when individual traders make mistakes.

Intuitively, the predictability arises for two reasons. First, *beliefs aggregation* dampens price fluctuations in markets with heterogeneous beliefs. The market price aggregates beliefs of traders by averaging their estimates using weights proportional to the square roots of precision parameters and not proportional to the precision parameters themselves. Jensen's inequality then implies that prices underreact to the total amount of private information available in the market. We discuss how this mechanism arises specifically when traders have correct beliefs about the error variances of their signals, an important conceptual issue in modeling information.

Second, an additional factor plays an important role in dynamic settings. In addition to placing long-term bets based on disagreement about the fundamental value of the asset, traders also engage in short-term trading based on how they believe other traders will revise their expectations in the future. Since this short-term speculation is based on beliefs about the dynamics of other traders' expectations and can result in traders taking positions opposite in sign to those implied by their own long-term valuations, it incorporates the logic of a *Keynesian beauty contest*.

Both effects tend to generate momentum in equilibrium returns. While the beliefs aggregation effect can arise in a one-period model, a Keynesian beauty contest intrinsically requires a dynamic model.

Rational Expectations Equilibrium. Our model highlights subtleties involved in defining important concepts such as a rational expectations equilibrium. There are two ways of thinking about the concept of rational expectations, which

we shall call *weak rational expectations* and *strong rational expectations*.

Weak rational expectations—equivalent to the efficient markets hypothesis—does not hypothesize that all traders are actually rational; instead, it hypothesizes that market prices aggregate information “as if” traders were rational. Hayek (1945) conjectured that markets aggregate information into the price “which might have been arrived at by one single mind possessing all the information which is in fact dispersed among all the people.” Muth (1961) defined rational expectations as market prices reflecting the “predictions of the relevant economic theory”; the subjective expectations of traders do not deviate systematically from the prediction of the relevant theory on average, but traders themselves may be irrational or make mistakes. Fama (1970) says that the efficient markets hypothesis is satisfied if market prices fully reflect information, “as if” traders are rational; the hypothesis does not require traders actually to be rational. LeRoy (1973) further recognizes that the concept of efficient markets requires a model of expected returns which rewards risk-taking appropriately. Lucas (1978) explicitly points out that the rational expectations hypothesis is not behavioral; it leaves aside how agents actually trade and think.

Strong rational expectations, by contrast, conjectures that all traders share a common prior and make rational decisions. Strong rational expectations can be interpreted as a behavioral model which conjectures that traders think and trade rationally. For example, Radner (1982) requires traders to share a common prior and apply Bayes law to learn from prices correctly; he says that in rational expectations equilibrium “the individual models are identical with the true model.” If rational behavior presumes sharing a common prior, it is well-known from the work of Grossman (1976) and Tirole (1982) that rational behavior results in an equilibrium with no speculative trade. If all traders think alike, there is no reason for any trader to acquire costly information and engage in speculative trading with intent to beat other traders. Therefore, strong rational expectations models have difficulty generating useful empirical hypotheses about trading in speculative markets which are often characterized by significant trading volume. When additional *ad hoc* ingredients are added to generate trade—such as noise trading, liquidity shocks, endowment shocks, shocks to private values, or behavioral biases—the result is a “noisy rational expectations” model.

Our model relies on differences in beliefs to generate both trade and return predictability. Models with agreement to disagree about hard-to-estimate parameters—such as the drift of a random dividend growth—are a realistic compromise between the strong rational expectations paradigm and approaches based on irrationality; such models are consistent with the weak rational expectation paradigm. Our model thus provides micro-foundations showing how almost-rational trading behavior may lead to predictable returns in equilibrium.

To motivate trade, we relax the common prior assumption in a minimal way. Traders are willing to trade because they symmetrically believe their private signals are more precise than their competitors believe them to be. Except for agreeing to

disagree about the precisions of their information, traders are otherwise completely rational. They apply Bayes Law consistently and optimize correctly. No additional behavioral assumptions or modeling ingredients, like noise trading, are needed to generate trade.

Our paper is consistent with Morris (1995), who eloquently argues for “dropping the common prior assumption from otherwise rational behavior” models as an important and largely overlooked modeling approach, since even rational agents may have heterogeneous beliefs. Empirical research, such as Barber and Odean (2001), also uses overconfidence.

Representative Agent and Economist. To think about predictability of returns in a structural model with different beliefs, we introduce two modeling devices, which we call a *representative agent* and an *economist*; both are characterized by their own sets of beliefs. Rubinstein (1975) explores what it means that “security prices fully reflect information” in markets with heterogenous beliefs and information along similar lines.

The first device is the representative agent. The representative agent, an artificial construct, is assumed to have possibly incorrect beliefs about model parameters which aggregate the information and beliefs of traders in such a way that the market clears at equilibrium prices. The representative agent has “the market’s beliefs.” Hirshleifer (1977) uses the term “representative agent” in the same way; Rubinstein (1975) refers to the representative agent’s beliefs as “consensus” beliefs. It is not *a priori* obvious that the representative agent can be characterized by a set of dogmatic beliefs about specific parameter values. Though they are not naive averages of beliefs of individual traders, one of our results is that the beliefs of the representative agent are closed-form functions of the parameters that describe the beliefs of traders in the model. We show that even when all of the traders in the model agree about the value of the innovation variance and mean reversion of an unobserved growth rate (and this agreement is common knowledge), the representative agent attaches different values to these parameters. The beliefs of the representative agent are hypothetical statistical constructs, not behavioral descriptions of the way traders actually think. It may therefore be dangerous, explicitly or implicitly, to attribute a behavioral bias to a representative agent, as in Daniel, Hirshleifer and Subrahmanyam (1998) or Barberis, Shleifer and Vishny (1998).

The representative agent is essentially a device which separates the testable pricing implications of a traditional asset pricing model from the testable quantity implications of a micro-founded model of trading behavior. The representative agent is concerned with making predictions about asset prices rather than about quantities traded, trading volume, and order flow.

The second device is the economist. The economist is an outsider to the model who is assumed to understand the structure of the model fully and to have empirically correct beliefs about model parameters. Muth (1961) takes a similar approach when defining a rational expectations equilibrium with an economist articulating

the relevant theory.

The economist and the representative agent do not necessarily have the same beliefs. The economist would believe that returns are unpredictable if and only if his beliefs happened to coincide with the representative agent's beliefs; this case would correspond to the efficient markets hypothesis. Except for some knife-edge cases, the economist and the representative agent in our model have different beliefs, and returns are predictable.

Vector Auto-Regressions. The model provides a formal economic underpinning for the extensive empirical literature that studies the predictability of returns at different horizons using past prices and dividends. In models with heterogeneous beliefs, equilibrium returns depend on both beliefs of the traders (aggregated in beliefs of the representative agent) and beliefs of the economist. We find that the expected return is a linear function of the following three state variables: (1) the difference between the market price and a valuation based only on the current dividend, i.e., the CARA-normal version of a valuation multiple based on earnings (or dividends); (2) an exponentially weighted historical average of this difference; and (3) an exponentially weighted historical average of dividend innovations. All three coefficients are usually non-zero.

Our model implies that the expected return is a linear function of state variables which follow a vector auto-regression (VAR). The state variables include the current levels of prices and dividends as well as exponentially weighted averages of past prices and dividends. The decay rates of past prices and dividends are proportional to the informativeness of prices, measured by the total precision in the market. Our model therefore places specific testable non-linear economic restrictions on VAR models of expected returns, discussed by Goyal and Welch (2003), Ang and Bekaert (2007), Cochrane (2008), Van Binsbergen and Koijen (2010), and Rytchkov (2012), among others. These restrictions are sufficiently flexible to be consistent with the patterns of short-term momentum and long-term mean-reversion. To reflect state variables appropriately, our approach suggests increasing the dimensionality of VAR systems by adding more lags, as in Campbell and Shiller (1988).

The derived complicated dynamics for stationary equilibrium returns suggests that a theoretical exploration of return predictability requires a fully dynamic infinite horizon model rather than a model with only two or three periods, such as Daniel, Hirshleifer and Subrahmanyam (1998) and Banerjee, Kaniel and Kremer (2009).

Our explanation for returns momentum differs from other explanations suggested in the previous literature. Return predictability in our paper is not related to changes in the aggregate amount of money chasing the return on the risky asset, as suggested by the research on flow-based predictability such as Gruber (1996), Lou (2012), and Vayanos and Woolley (2013). Market clearing implies that the aggregate flow of money into the market for risky assets is zero, even though individual traders indeed find profitable investment opportunities and chase returns

while the economist finds anomalies.

Return predictability in our paper is also not related to noisy aggregation of heterogeneous information. Since the model is symmetric and there is no noise trading, the price reveals a sufficient statistic for what each trader cares to know about other traders' private information. The equilibrium price averages traders' expectations, calculated under different beliefs but with the same information set. This is different from Allen, Morris and Shin (2006), who show that aggregation of noisy heterogeneous information, even in settings with a common prior, can lead to *ex post* price drift. It is also different from Banerjee, Kaniel and Kremer (2009), who argue that asymmetric information alone cannot generate momentum; they claim instead that agreement to disagree about the average valuation is necessary for heterogeneous beliefs to generate momentum. In contrast to both of these papers, we obtain return predictability in a model in which traders who are correct on average infer from prices a noiseless sufficient statistic for the private information of others and agree about the average valuation, even though they disagree about one another's current and future valuations.

It is fashionable to attribute predictability in asset returns to irrational behavior motivated by psychology. This presumes that rational behavior—not motivated by psychology—would lead to no return predictability. Simon (1957) proposes the concept of bounded rationality for studying the irrationality of human choices resulting from various institutional constraints such as the psychological costs of acquiring information, cognitive limitations of human minds, or the finite amount of time humans have to make a decision. For example, Daniel, Hirshleifer and Subrahmanyam (1998) have to assume that the representative agent exhibits a biased self-attribution leading to time-varying overconfidence. Hong and Stein (1999), Barberis and Shleifer (2003), and Greenwood and Shleifer (2014) assume that traders follow simple trading rules and do not extract information from prices. When return anomalies are motivated by behavioral biases, Fama (1998) suggests that a Pandora's box is opened, undermining modeling parsimony by enabling one plethora of behavioral biases to explain another plethora of anomalies.

Our approach not only allows us to generate momentum in returns, but its predictions are also consistent with empirical findings on momentum patterns. Indeed, Lee and Swaminathan (2000) and Cremers and Pareek (2014) find that momentum is stronger for stocks with higher trading volume and short-term trading, respectively. Moskowitz, Ooi and Pedersen (2012) find that more liquid contracts in equity index, currency, commodity, and bond futures markets tend to exhibit greater momentum profits. Zhang (2006) and Verardo (2009) show that momentum returns are larger for stocks with higher analysts' disagreement. Similar properties characterize momentum patterns in our model.

The information structure in our model is similar to Kyle and Lin (2002), Scheinkman and Xiong (2003), and Kyle, Obizhaeva and Wang (2016). Our paper differs from the last one in that it has competitive trading rather than strategic trading. The assumption of perfect competition allows us to prove most of our results analytically.

ically. The assumptions of zero-net-supply and a constant absolute risk aversion approximate markets for individual stocks, where risks are idiosyncratic and wealth effects are not significant. This differs from papers which focus on the interaction between beliefs aggregation and wealth effects but leave aside private information, such as Detemple and Murthy (1994), Basak (2005), Jouini and Napp (2007), Xiong and Yan (2010), Cujean and Hasler (2014), and Atmaz and Basak (2015). In our model, the beliefs of the representative agent do not vary with the distribution of wealth among traders, and these beliefs are consistent with the Bayes Law.

Conceptually, our approach is most similar to the approach of Campbell and Kyle (1993), who use noise trading to generate excess volatility and mean reversion instead of relative overconfidence to generate momentum.

Plan. This paper is structured as follows. Section 1 discusses stylized examples illustrating how the market’s incorrect beliefs can explain anomalies. Section 2 presents the model. Section 3 explains the two dampening effects. Section 4 discusses how momentum can arise in a model with heterogenous beliefs about private information and shows that the beliefs of the representative agent are not simply “averages” of traders’ beliefs. Section 5 analyzes holding-period returns as functions of the empirically correct beliefs of the economist and possibly incorrect beliefs of traders. Section 6 concludes. All proofs are in the Appendix.

1. Motivating Examples

We motivate our discussion with three examples in which the beliefs of the representative agent reflect the beliefs of the market and the beliefs of the economist reflect empirically correct beliefs. All examples illustrate how return predictability results when market beliefs deviate from empirically correct beliefs. None of the examples provides intuition of why market beliefs may differ from empirically correct beliefs. In the following section, we will show how aggregation of dynamic trading decisions of individual market participants with different beliefs about private information can naturally lead to distortions in market’s beliefs.

These examples illustrate several important principles:

- The actual return process depends on two sets of parameters: the empirically correct parameters and possibly incorrect parameters used by the market.
- The possibly incorrect parameters used by the market affect the expected return, return volatility, and the holding-period return over different horizons.
- It is usually more appropriate to model financial markets using dynamic steady-state models because the insights of static non-stationary models often cannot be easily mapped into data.

While none of these examples corresponds precisely to the model examined in the paper, they are helpful for understanding its main point: Realistic micro-founded modeling of return dynamics requires a dynamic setting in which returns

are influenced both by correct parameter values and traders' possibly incorrect beliefs about them.

1.1. The Gordon Growth Model With Geometric Brownian Motion Dividends.

The simplest illustration assumes that the market (representative agent) uses a possibly incorrect dividend growth rate when applying the Gordon growth model to an asset whose dividend follows a geometric Brownian motion process

$$(1) \quad dD(t) = \check{\gamma} D(t) dt + \sigma D(t) dB(t).$$

Here, $D(t)$ is the dividend rate at time t , $\check{\gamma}$ is the constant growth rate the representative agent expects, σ is the volatility of dividends, and $B(t)$ is a standardized Brownian motion.

Throughout this paper, a “breve” (“ $\check{}$ ”) indicates a possibly empirically incorrect parameter value assigned by the representative agent, and a “hat” (“ $\hat{}$ ”) indicates an empirically correct parameter value assigned by the economist. The representative agent and the economist agree about parameters without “breves” or “hats”. Let $\hat{E}_t\{\dots\}$ and $\hat{\text{Var}}_t\{\dots\}$ denote expectation and variance operators calculated using information available at time t based on empirically correct beliefs of the economist.

Suppose that the market requires expected return r . Then a simple application of the Gordon growth formula yields the market price

$$(2) \quad P(t) = \frac{D(t)}{r - \check{\gamma}}.$$

The market believes the actual percentage return process is

$$(3) \quad \frac{dP(t) + D(t) dt}{P(t)} = r dt + \sigma dB(t).$$

The market's expected return r can be decomposed into a return of $r - \check{\gamma}$ from the dividend yield $D(t)/P(t)$ and an expected return of $\check{\gamma}$ from capital gains $dP(t)/P(t)$.

Suppose the market beliefs are possibly incorrect, and the empirically correct growth rate in equation (1) is $\hat{\gamma}$, not $\check{\gamma}$. Then the actual expected return is given by

$$(4) \quad \hat{E}_t \left\{ \frac{dP(t) + D(t) dt}{P(t) dt} \right\} = r - \check{\gamma} + \hat{\gamma}.$$

When $\check{\gamma} = \hat{\gamma}$, the actual expected return is equal to r . Otherwise, the market obtains an expected return of $r - \check{\gamma} + \hat{\gamma}$; the observed dividend yield $r - \check{\gamma}$ remains unchanged, but the unobserved expected return from capital gains changes from $\check{\gamma}$

to $\hat{\gamma}$. This example illustrates that the actual expected return $r - \check{\gamma} + \hat{\gamma}$ depends on two parameters: the market's expected growth rate $\check{\gamma}$ and the empirically correct expected growth rate $\hat{\gamma}$. When the market has a more pessimistic expected growth rate $\check{\gamma}$, this increases the dividend yield by making the asset cheap and therefore raises the expected return.

In this example, both the expected return and volatility σ are constant over time. The volatility is not affected by the market's expectations of the growth rate. In the next example, the expected return varies over time, and the constant standard deviation of the dollar return is a function of the market's beliefs about parameters governing the dividend process.

1.2. Excess Volatility and Mean Reversion With Arithmetic AR-1 Dividends.

Suppose the representative agent believes that de-measured dividends follow an Ornstein-Uhlenbeck process given by

$$(5) \quad dD(t) = -\check{\alpha} (D(t) - \bar{D}) dt + \sigma dB(t),$$

where $D(t)$ is the dividend rate, $\check{\alpha}$ is the market's belief about the constant rate of mean reversion, σ is the volatility of dividends, \bar{D} is the constant steady-state mean dividend level, and $B(t)$ is a standardized Brownian motion. Assume that the required rate of return is the risk-free rate r , consistent with a zero-net-supply asset. Then the asset's price $P(t)$ is given by

$$(6) \quad P(t) = \frac{\bar{D}}{r} + \frac{D(t) - \bar{D}}{r + \check{\alpha}}.$$

This formula is obtained by applying the Gordon growth formula separately to the two components \bar{D} and $D(t) - \bar{D}$, with growth rates of zero and $-\check{\alpha}$, respectively.

Suppose that the representative agent's beliefs about the mean-reversion parameter in equation (5) are possibly incorrect, and the correct value of the mean-reversion parameter is $\hat{\alpha}$, not $\check{\alpha}$. The correct return process (in dollars per share) is given by

$$(7) \quad dP(t) + D(t) dt = r P(t) dt + \frac{\check{\alpha} - \hat{\alpha}}{r + \check{\alpha}} (D(t) - \bar{D}) dt + \frac{\sigma}{r + \check{\alpha}} dB(t).$$

The empirically correct expected dollar return per share is given by

$$(8) \quad \hat{E}_t \left\{ \frac{dP(t) + D(t) dt}{dt} \right\} = r P(t) + \frac{\check{\alpha} - \hat{\alpha}}{r + \check{\alpha}} (D(t) - \bar{D}).$$

The market obtains an expected dollar return $r P(t)$ when $\check{\alpha} = \hat{\alpha}$; otherwise, the market also obtains a time-varying unexpected excess dollar return per share $(\check{\alpha} - \hat{\alpha}) (r + \check{\alpha})^{-1} (D(t) - \bar{D})$.

The representative agent's beliefs also affect the volatility of returns. The standard deviation of the dollar return per share is

$$(9) \quad \widehat{\text{Var}}_t^{1/2} \left\{ \frac{dP(t) + D(t) dt}{dt^{1/2}} \right\} = \frac{\sigma}{r + \check{\alpha}}.$$

The volatility of the return depends on the market's possibly incorrect mean-reversion parameter $\check{\alpha}$, not on the empirically correct mean-reversion parameter $\hat{\alpha}$.

If the representative agent believes that the dividend process is more persistent than it actually is—i.e., $\check{\alpha} < \hat{\alpha}$ —then there is excess volatility and mean reversion. There is excess volatility because the actual volatility $\sigma (r + \check{\alpha})^{-1}$ is greater than the volatility $\sigma (r + \hat{\alpha})^{-1}$ that would be obtained if the market used the correct mean-reversion rate $\hat{\alpha}$. There is mean reversion because the expected excess return $(\check{\alpha} - \hat{\alpha}) (r + \check{\alpha})^{-1} (D(t) - \bar{D})$ is negative (positive) when dividends and therefore prices are above (below) their long-term mean. It can be shown that the entire term structure of the expected holding-period return varies over time as well.

1.3. A Two-Period Model With Information Processing.

Prices reflect the way in which markets process information, perhaps correctly or perhaps incorrectly. Our third example shows that when market prices reflect information which is processed incorrectly, this may lead to return predictability.

Consider the following two-period model. Suppose a risky asset has an unobserved liquidation value v . The market observes a signal denoted ΔI and believes that the signal has the form $\Delta I = \check{\tau}^{1/2}v + z$, where $\check{\tau}$ is the market's possibly incorrect belief about the precision of the signal. The random variables v and z are identically and independently distributed as $N(0, 1)$. The variable ΔI has a simple signal-plus-noise form. The initial price P_0 is normalized to zero at time $t = 0$. Upon observation of the signal at time $t = 1$, the market's expectation of the asset's liquidation value changes to P_1 . At time 2, the liquidation value v is realized. The empirically correct value $\hat{\tau}$ of the precision parameter is possibly different from the market's belief $\check{\tau}$.

The two periods in this simple model are quite different. Assuming no discounting, the expected return and price volatility over the period from $t = 0$ to $t = 1$ are given by

$$(10) \quad \hat{\text{E}}\{P_1 - P_0 \mid \Delta I\} = \frac{\check{\tau}^{1/2}}{1 + \check{\tau}} \Delta I, \quad \widehat{\text{Var}}^{1/2}\{P_1 - P_0 \mid \Delta I\} = \frac{\check{\tau}^{1/2}(1 + \hat{\tau})^{1/2}}{1 + \check{\tau}}.$$

In contrast, over the period from $t = 1$ to $t = 2$, the expected return and price volatility are given by

$$(11) \quad \hat{\text{E}}\{v - P_1 \mid \Delta I\} = \left(\frac{\hat{\tau}^{1/2}}{1 + \hat{\tau}} - \frac{\check{\tau}^{1/2}}{1 + \check{\tau}} \right) \Delta I, \quad \widehat{\text{Var}}^{1/2}\{v - P_1 \mid \Delta I\} = \frac{\left((1 + \check{\tau} - \hat{\tau}^{1/2}\check{\tau}^{1/2})^2 + \check{\tau} \right)^{1/2}}{1 + \check{\tau}}.$$

If the market's beliefs are correct ($\check{\tau} = \hat{\tau}$), then the expected return for the second period is zero, and return variances during the two periods are $\hat{\tau} (1 + \hat{\tau})^{-1}$ and $(1 + \hat{\tau})^{-1}$, respectively. Otherwise, various patterns in the expected return and volatility are possible depending on particular sets of parameters $\check{\tau}$ and $\hat{\tau}$.

The predictions are quite different for the two periods. For example, for different parameter values, the first-period volatility may be lower or higher than the second-period volatility. It is difficult to infer what these discrete-time results imply for intrinsically dynamic pricing anomalies related to volatility in stationary dynamic models. Any one-period, two-period, or three-period model, including Daniel, Hirshleifer and Subrahmanyam (1998), faces the same difficulty.

1.4. Summary of Motivating Examples.

The three motivating examples are all based on modeling market prices as the result of a single representative agent processing information. The first example shows that overly pessimistic beliefs about the growth rate of dividends lead to a higher expected return, thus providing an explanation for the equity premium puzzle of Mehra and Prescott (1985). The second example shows that a belief that a mean-reverting dividend process is more persistent than implied by the actual rate of mean reversion leads to excess volatility and mean reversion in asset prices, consistent with Shiller (1981). The third example shows that overconfidence about the precisions of signals can lead to excess volatility and mean reversion. Overconfidence increases the sensitivity of price changes to information; this makes the risk premium counter-cyclical, consistent with Campbell and Shiller (1988) and Fama and French (1989). The intrinsic limitations of a two-period model remind us that dynamic steady-state models are more appropriate for studying return dynamics.

All three of these motivating examples generate returns predictability by assuming that the market's beliefs are different from the beliefs of an economist who knows the correct parameter values. In what follows, we address the challenging problem of generating return predictability in a model in which the traders have different beliefs but their beliefs agree, on average, with the beliefs of the economist. Next, we present a dynamic, continuous-time model in which we show that market beliefs deviate from empirically correct beliefs due to the way in which beliefs about private information are aggregated.

2. A Competitive Model With Disagreement and Information Processing

To examine return predictability when markets aggregate traders' heterogeneous beliefs about privately observed information, we present an intuitively realistic model of how traders think and trade.

Given their individual beliefs, traders behave in a rational manner. They collect public and private information, construct signals from the information, and use

the signals to predict asset returns. In doing so, traders apply Bayes Law correctly and calculate target positions based on what their signals tell them. Although each individual trader behaves rationally, the model exhibits collective irrationality in that each trader is relatively overconfident, believing that the precision of his own private information flow is greater than other traders believe it to be. Without this element of collective irrationality, it would be difficult to construct a model in which trade occurs without adding noise traders, liquidity traders, or other traders who trade expecting to lose money.

2.1. Model Assumptions

There are N risk-averse competitive traders who trade at price $P(t)$ a risky asset in zero net supply against a risk-free asset which earns constant risk-free rate $r > 0$.

The risky asset pays out dividends at continuous rate $D(t)$. Dividends follow a stochastic process with mean-reverting stochastic growth rate $G^*(t)$, constant instantaneous volatility $\sigma_D > 0$, and constant rate of mean reversion $\alpha_D > 0$:

$$(12) \quad dD(t) := -\alpha_D D(t) dt + G^*(t) dt + \sigma_D dB_D(t).$$

The dividend $D(t)$ is publicly observable, but the growth rate $G^*(t)$ is not observed by any trader. The growth rate $G^*(t)$ follows an AR-1 process with the mean-reversion α_G and volatility σ_G :

$$(13) \quad dG^*(t) := -\alpha_G G^*(t) dt + \sigma_G dB_G(t).$$

If both the dividend $D(t)$ and $G^*(t)$ were observable, then the price of the asset would equal its fundamental value given by the generalization of the Gordon growth formula

$$(14) \quad F(t) = \frac{D(t)}{r + \alpha_D} + \frac{G^*(t)}{(r + \alpha_D)(r + \alpha_G)}.$$

Each trader observes public and private signals about the growth rate $G^*(t)$, then constructs an estimate of the fundamental value $F(t)$ by replacing $G^*(t)$ in equation (14) with its expectation.

For all dates $t > -\infty$, each trader n chooses consumption $c_n(t)$ and inventories of the risky asset $S_n(t)$ to maximize an expected constant-absolute-risk-aversion (CARA) utility function $U(c_n(s)) := -e^{-A c_n(s)}$ with risk aversion parameter A . Letting $\rho > 0$ denote a time preference parameter, trader n solves the maximization problem

$$(15) \quad \max_{\{c_n, S_n\}} \mathbf{E}_t^n \left\{ \int_{s=t}^{\infty} e^{-\rho(s-t)} U(c_n(s)) ds \right\},$$

where the wealth $W_n(t)$ follows the process

$$(16) \quad dW_n(t) = r W_n(t) dt + S_n(t) (dP(t) + D(t) dt - r P(t) dt) - c_n(t) dt.$$

Each trader takes prices in equation (16) as given.

We use $E_t^n\{\dots\}$ to denote the expectation of trader n calculated with respect to his information at time t , which consists of both private information as well as public information extracted from the history of dividends and prices, as discussed below. The information structure is the same as the smooth-trading model of Kyle, Obizhaeva and Wang (2016).

Let $G_n(t) := E_t^n\{G^*(t)\}$ denote trader n 's estimate of the growth rate. Let Ω denote the steady state error variance of the estimate of $G^*(t)$, scaled in units of the standard deviation of its innovation σ_G :

$$(17) \quad \Omega := \text{Var} \left\{ \frac{G^*(t) - G_n(t)}{\sigma_G} \right\}.$$

If time is measured in years, for example, $\Omega = 4$ has the interpretation that the estimate of $G^*(t)$ is “behind” the true value of $G^*(t)$ by an amount equivalent to four years of volatility unfolding at rate σ_G per year.

Each trader n observes a continuous stream of private information $I_n(t)$ about the scaled unobservable growth rate $G^*(t)$:

$$(18) \quad dI_n(t) := \tau_n^{1/2} \frac{G^*(t)}{\sigma_G \Omega^{1/2}} dt + dB_n(t).$$

Each trader is certain that his own private information $I_n(t)$ has high precision $\tau_n = \tau_H$ and the other traders' private information has low precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$.

Since the equilibrium price reveals the average signal in the symmetric model, each trader infers the average of other traders' private signals from the market price.

Each trader also infers information $I_0(t)$ about the growth rate from the dividend stream $D(t)$. To simplify notation for the analysis of the information content of dividends, define $dI_0(t) := (\alpha_D D(t) dt + dD(t))/\sigma_D$ with $dB_0 := dB_D$ and

$$(19) \quad \tau_0 := \Omega \sigma_G^2 / \sigma_D^2.$$

Then the process

$$(20) \quad dI_0(t) := \tau_0^{1/2} \frac{G^*(t)}{\sigma_G \Omega^{1/2}} dt + dB_0(t)$$

is informationally equivalent to the process $D(t)$, where $dB_0, dB_G, dB_1, \dots, dB_N$ are independent Brownian motions.

Since its drift is proportional to $G^*(t)$, each increment $dI_n(t)$ in equation (18) is a noisy observation of the unobserved growth rate $G^*(t)$. In equations (18) and (20), the parameter $\sigma_G \Omega^{1/2}$ is a scaling coefficient, which scales τ_n so that $\tau_n dt$ is the R^2 of the predictive regression of $G^*(t) - G_n(t)$ on $dI_n(t)$. This is a convenient

way to model information flow because the precision parameter τ_n measures the informativeness of the signal $dI_n(t)$ as a signal-to-noise ratio describing how fast the information flow generates a signal of a given level of statistical significance.

Agreement to disagree—a realistic compromise between rational models and behavioral finance models—is the mechanism that generates trade in our model. Traders believe that they can make profits at the expense of others, even though it is common knowledge that aggregate profits are equal to zero. Traders agree on the precision τ_0 of public information and agree to disagree about the precision of private information. It is a common knowledge that each trader believes his own signal has high precision τ_H while signals of the others have low precision τ_L . Symmetry implies that traders agree on the total precision

$$(21) \quad \tau := \tau_0 + \tau_H + (N - 1) \tau_L.$$

Note also that by construction all traders agree about the variance of information flow in (18) and (20). It would be inappropriate to have traders disagree about variances of diffusion processes, since they can be estimated as precisely as necessary by observing them continuously.

The model is not consistent with the Bayesian Nash equilibrium concept of Harsanyi because traders' beliefs about precision parameters are inconsistent with a common prior distribution. According to Harsanyi's approach, each trader's own type—characterizing his preferences and beliefs—is drawn randomly from a set of possible types at the beginning of the extended game, and each trader updates his beliefs using Bayes law; traders know their own type and share a common prior, i.e., they all “agree” about the structure of the game. In models with agreement to disagree, traders do not share a common prior, but each trader does apply Bayes law consistently.

Given N , the parameters α_G , σ_G , τ_H , and τ_L describe the traders' belief structures concerning information about the unobserved growth rate. Due to symmetry, these belief structures imply the same values of Ω , τ_0 , and τ for all traders. The entire structure of the model is common knowledge. Traders agree about all parameter values, except that traders symmetrically agree to disagree about the precisions τ_H and τ_L of their own and other traders' signals.

2.2. Model Solution

Stratonovich-Kalman-Bucy filtering implies that the steady-state error variance is given by

$$(22) \quad \Omega := \text{Var} \left\{ \frac{G^*(t) - G_n(t)}{\sigma_G} \right\} = \frac{1}{2 \alpha_G + \tau}.$$

Trader n 's estimate $G_n(t)$ can be conveniently written as the weighted sum of three sufficient statistics $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$, which summarize the information content of dividends, the trader's private information, and other traders'

private information, respectively. Define

$$(23) \quad H_n(t) := \int_{u=-\infty}^t e^{-(\alpha_G + \tau)(t-u)} dI_n(u), \quad n = 0, 1, \dots, N,$$

and

$$(24) \quad H_{-n}(t) := \frac{1}{N-1} \sum_{m=1, \dots, N, m \neq n} H_m(t).$$

These formulas have an intuitive interpretation. The importance of each bit of information dI_n about the growth rate decays exponentially at a rate $\alpha_G + \tau$, i.e., the sum of the natural decay rate of fundamentals α_G and the speed at which the others learn about fundamentals τ .

The filtering formulas further imply that trader n 's expected growth rate is

$$(25) \quad G_n(t) := \sigma_G \Omega^{1/2} \left(\tau_0^{1/2} H_0(t) + \tau_H^{1/2} H_n(t) + (N-1) \tau_L^{1/2} H_{-n}(t) \right).$$

When forming his estimate, each trader assigns a larger weight $\tau_H^{1/2}$ to his own signal and a smaller weight $\tau_L^{1/2}$ to each of the other traders' signals. Trade occurs as a result of the different weights used by traders.

Each trader calculates a target inventory proportional to his risk tolerance and the difference between his own valuation and the average valuation of other traders. The following theorem characterizes equilibrium for the continuous-time model with perfect competition.

THEOREM 1: *There exists a steady-state Bayesian-perfect equilibrium with symmetric linear strategies and with positive trading volume if and only if the three polynomial equations (A-19)–(A-21) have a solution, and traders' demand curves are downward sloping. Such an equilibrium has the following properties:*

- 1) *There is an endogenously determined constant $C_L > 0$, defined in equation (A-12), such that trader n 's optimal inventories $S_n(t)$ are*

$$(26) \quad S_n(t) = C_L (H_n(t) - H_{-n}(t)).$$

- 2) *There is an endogenously determined constant $C_G > 0$, defined in equation (A-10), such that the equilibrium price is*

$$(27) \quad P(t) = \frac{D(t)}{r + \alpha_D} + C_G \frac{\bar{G}(t)}{(r + \alpha_D)(r + \alpha_G)},$$

where $\bar{G}(t) := \frac{1}{N} \sum_{n=1}^N G_n(t)$ denotes the average of traders' expected growth rates.

Formula (27) is similar to the Gordon growth formula in equation (2) in the first motivating example and equation (6) in the second motivating example, with two important exceptions. First, the growth rate $\bar{G}(t)$ is the average of traders' expected growth rates, not a particular trader's growth rate. Second, the Gordon growth formula would imply that $C_G = 1$, but we will show $C_G < 1$ below.

The competitive equilibrium here is very different from the imperfectly competitive smooth-trading equilibrium of Kyle, Obizhaeva and Wang (2016). The most important difference is that competitive traders do not smooth their trading out over time but instead immediately adjust inventories to levels equal to the target inventory $C_L (H_n(t) - H_{-n}(t))$.

In the symmetric equilibrium, the price instantly and fully reveals all information $\sum_{n=1}^N H_n(t)$. From equation (25), it is straightforward to show that the equilibrium price (27) can be written as

$$(28) \quad P(t) = \frac{D(t)}{r + \alpha_D} + C_G \frac{\sigma_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} H(t),$$

where the weighted-average signal $H(t)$ is defined as

$$(29) \quad H(t) = \tau_0^{1/2} H_0(t) + \tau_I^{1/2} \sum_{n=1}^N H_n(t)$$

and parameter τ_I is defined as

$$(30) \quad \tau_I^{1/2} := \frac{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}{N}.$$

The parameter τ_I essentially plays a role of “implied” symmetric beliefs, except that the implied error variance is not consistent with the definition of Ω in this equation. Even though the price instantly reveals all information, we will show next that there is time-series momentum.

3. Price Dampening

The way in which prices average dynamic information with disagreement tends to dampen price fluctuations and lead to time-series momentum in returns. In this section, we explain why this occurs.

3.1. Intuition Behind Two Dampening Effects

To better explain the intuition behind momentum patterns, we plug the estimates of the growth rates (25) into the equilibrium price (27) and write it in a slightly different form as

$$(31) \quad P(t) = \frac{D(t)}{r + \alpha_D} + \frac{C_G \sigma_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \left(\tau_0^{1/2} H_0(t) + C_J \left(\frac{1}{N} \tau_H + \frac{N-1}{N} \tau_L \right)^{1/2} \sum_{n=1}^N H_n(t) \right).$$

Here, the constant C_J denotes the ratio of the average of the square roots to the square root of the average of precisions:

$$(32) \quad C_J := \left(\frac{1}{N} \tau_H^{1/2} + \frac{N-1}{N} \tau_L^{1/2} \right) \left(\frac{1}{N} \tau_H + \frac{N-1}{N} \tau_L \right)^{-1/2}.$$

If $C_G = C_J = 1$, then equation (31) implies that prices are equal to the expected fundamental value of the asset as if all information in the market—both public and private—were included into the information set. The prices are described by the Gordon growth formula with the estimate of the growth rate equal to the weighted sum of a signal $H_0(t)$ and a signal $\sum_{n=1}^N H_n(t)$ with precisions of τ_0 and $\frac{1}{N}(\tau_H + (N-1)\tau_L)$, respectively. This is a full-information benchmark. It assumes that the precision of each signal is the average of traders' different beliefs about its precision. When the two constants C_G and C_J are both equal to one, there is no returns momentum when the correct empirical precisions of the signals are equal to the average $\frac{1}{N} \tau_H + \frac{N-1}{N} \tau_L$.

When $C_G \neq 1$ or $C_J \neq 1$, returns are generally predictable. The following proposition states the important result that the constants C_G and C_J in equation (31) are usually less than one, thus dampening equilibrium prices and leading to momentum.

PROPOSITION 1: *Assume that traders are correct on average in the sense that the empirically correct precision of private signals is $\frac{1}{N} \tau_H + \frac{N-1}{N} \tau_L$. If $\tau_H = \tau_L$, then $C_J = 1$ and $C_G = 1$; the expected return is equal to the risk-free rate, and there is no dampening effect. If $\tau_H > \tau_L$, implying traders are relatively overconfident, then*

$$(33) \quad 0 < C_J < 1 \quad \text{and} \quad 0 < C_G \leq \left(1 + \frac{N-1}{N} \frac{\left(\tau_H^{1/2} - \tau_L^{1/2} \right)^2}{r + \alpha_G} \right)^{-1} < 1,$$

implying prices are “dampened” relative to a full-information benchmark, and this is associated with returns momentum.

As disagreement decreases, both constants C_G and C_J converge to one, and momentum goes away. The proof is presented in Appendix A.2. Next we discuss the intuition for this result.

First, the constant $C_J \leq 1$ governs the weights placed on signals when traders' expectations are averaged into the market price. Since the price is fully revealing, traders' expectations are different because the traders have different beliefs, not because they have different information; therefore, C_J measures *beliefs aggregation*. When traders are overconfident ($\tau_H > \tau_L$), Jensen's inequality implies $C_J < 1$.

The equilibrium price (27) or (31) averages valuations of traders with estimates of the growth rate (25) that reflect traders' information using weights proportional to the square roots of precisions. Averaging beliefs across traders leads to price

dampening since the average of square roots is less than the square root of the average. Disagreement about the precision of private information makes the average valuation less sensitive to aggregate information in comparison to a full-information benchmark.

Second, the endogenous constant $C_G \leq 1$ reflects a *Keyensian beauty contest*. In the competitive model with overconfident traders, we formally prove in Proposition 1 that C_G satisfies $0 < C_G < 1$ when $\tau_H > \tau_L$ and $C_G = 1$ when $\tau_H = \tau_L$. This endogenously determined coefficient C_G makes equations (27) and (31) differ from the average valuation of traders so that the market price is less sensitive to changes in the average growth rate estimate of traders than if it were defined by applying the Gordon growth formula. This effect is a result of short-term speculative trading based on a specific endogenous dynamics of disagreement about a common value of a growth rate. Each trader disagrees with others about how to interpret private information. He also expects others to learn about their mistakes and revise their valuations in the short-term future, yet ultimately converging in the direction of his own valuation in the long run. Since each trader may expect other traders to revise their expectations in the “wrong” direction in the short run, the trader will attempt to profit from these adjustments by trading ahead of them, even if this means trading against his own long-term valuation. We provide a formal analysis of expectations dynamics in Appendix A.3. This short-term trading due to the endogenous Keynesian beauty contest dampens prices relative to the average fundamental valuation in the market.

The mechanism in our model is entirely different from the mechanism in Allen, Morris and Shin (2006), where prices are also not equal to the expectation of fundamentals under a full-information benchmark because public information tends to be over-weighted relative to private information. In their model, traders share a common prior, they learn about the average private signal in the presence of noise trading, and the price reacts sluggishly to changes in private information, thus creating an impression of momentum in the realized price paths ex post. Their mechanism based on noisy prices is unrelated to the beliefs aggregation and Keynesian beauty contest in our model. In our symmetric model, all information is fully revealed at any moment of time; prices have a non-zero drift even though there is no noise in prices.

Conceptual Point Related to Modeling Private Information.

Our results relate to an important conceptual point about how to model information in a market microstructure setting. Whether in a static or dynamic setting, beliefs aggregation depends crucially on how information is scaled. We illustrate this point in Appendix A.4 using a simple one-period model which is similar to our dynamic model.

Competitive traders trade a risky asset with a liquidation value $v \sim N(0, 1/\tau_v)$. All traders obtain public and private information. Traders agree to disagree about the precision of private signals. Information is scaled in two ways.

In the first case, information is modeled—as in most microstructure papers—as $v + \epsilon$ with $v \sim N(0, \tau_v^{-1})$ and $\epsilon \sim N(0, \tau_\epsilon^{-1})$; traders disagree about the value of the parameter τ_ϵ , i.e., more precise information is modeled as a lower variance of the noise component. The variance of the signal $v + \epsilon$ is $\tau_v^{-1} + \tau_\epsilon^{-1}$, and traders disagree about this variance.

In the second case, information is modeled as $\tau_n^{1/2}v + \epsilon$ with $v \sim N(0, \tau_v^{-1})$ and $\epsilon \sim N(0, 1)$; traders disagree about parameter τ_n ; more precise information is modeled as a larger weight assigned to signals. Each trader believes his private signal has precision τ_H and other private signals have precisions τ_L with $\tau_H > \tau_L$. The variance of the signal $\tau_n^{1/2}v + \epsilon$ is $\tau_n \tau_v^{-1} + 1$, and traders disagree about this variance as well.

The only difference between the two models concerns the manner in which information is scaled. When traders share a common prior, the scaling of information does not matter because the information can be re-scaled by multiplying it by an appropriate constant. When traders do not share a common prior, they disagree about the appropriate scaling constant.

The equilibrium prices in these two cases have strikingly different properties.

In the first case, the equilibrium price is equal to the expectation of a fundamental value as if all public and private information were included into information set; intuitively, information i_0 is assigned precision τ_0 , each private information i_n is assigned the average precision $\frac{1}{N}(\tau_H + (N - 1)\tau_L)$. In this model, there is no C_J effect. This one-period model always generates $C_J = 1$.

In the second case, the equilibrium price can be thought of as the expectation of a fundamental value in a full-information case as well; whereas information i_0 is still assigned precision τ_0 , private information i_n is not assigned the average precision, but instead obtains the precision $(\frac{1}{N}(\tau_H^{1/2} + (N - 1)\tau_L^{1/2}))^2$. Due to Jensen's inequality, this imputed precision is lower than the average precision. This is the same mechanism generating beliefs aggregation in our continuous-time model; it effectively implies $C_J < 1$.

Which of these two ways of modeling private information is preferable? We believe the second way of modeling information is preferable because it is consistent with a dynamic setting. The noise term of a discrete signal naturally maps into a diffusion term in information flow, whereas the signal term maps into its drift. The first approach results in a dynamic signal like $v\Delta t + \tau_n^{-1/2}\Delta Z$ with $\text{Var}\{\Delta Z\} = \Delta t$. The second approach results in a dynamic signal like $\tau_n^{1/2}v\Delta t + \Delta Z$. In the first approach, the diffusion variance of the signal per unit of time is $\tau_v^{-1}\Delta t + \tau_n^{-1} \rightarrow \tau_n^{-1}$ as $\Delta t \rightarrow 0$; traders disagree about this variance because they disagree about τ_n . In the second approach, the diffusion variance of the signal per unit of time is $\tau_n\tau_v^{-1}\Delta t + 1 \rightarrow 1$ as $\Delta t \rightarrow 0$; traders agree that the diffusion variance of the signal is one, the variance of a standardized Brownian motion. Our continuous-time model is consistent with this second approach because it maps directly into equations (18) and (20). In a continuous-time model, a trader can infer the diffusion variance with high accuracy by observing the information process over short periods

of time. Therefore, it does not make sense for one trader to assume that another trader observes the diffusion variance of his signal incorrectly.

We conclude that, taken to a continuous-time setting, the first approach is not consistent with minimal rationality; therefore, the second approach, consistent with the continuous-time model in this paper, is the correct one.

If we think of a one-period model as a story about a dynamic model, then it is appropriate to assume that traders should not be modeled as disagreeing about the variance of the noise term in a one-period model either. To illustrate this, assume that the ratio τ_H/τ_L is a somewhat large number, say $\tau_H/\tau_L = 100$. Suppose a trader believes himself to have precision τ_H with probability 0.9999 and τ_L with probability 0.0001. Suppose the trader observes a signal which is plus-or-minus one standard deviation from its mean under the assumption the trader has precision τ_L . Under the assumption that the trader has a precision τ_H , the same signal is approximately plus-or-minus ten standard deviations from its mean. Since the probability of a ten standard deviation event is virtually zero, the trader would revise his estimate of having a high-precision from 0.9999 down to approximately zero. More realistically, traders are likely to standardize signals so that their variances are equal to one. This effectively implies that the second modeling approach is the correct one.

3.2. Properties of Momentum

The equilibrium prices are dampened relative to the estimate of fundamental value due to $C_G < 1$ and $C_J < 1$, and therefore returns exhibit momentum. We will show that, consistent with empirical evidence, these momentum effects are more pronounced when the degree of disagreement is larger, markets are more liquid, and trading volume is more substantial.

We start by discussing several properties of the equilibrium.

First, the market tends to be more liquid when there is more disagreement. Define λ as

$$(34) \quad \lambda := \frac{C_G \sigma_G \Omega^{1/2} \tau_I^{1/2}}{(r + \alpha_D)(r + \alpha_G) C_L}.$$

Then, using equations (26) and (27), the equilibrium price can be written as

$$(35) \quad P(t) = \frac{D(t)}{r + \alpha_D} + \lambda \frac{C_L}{\tau_I^{1/2}} \left(\tau_0^{1/2} H_0(t) + \tau_I^{1/2} N H_{-n}(t) \right) + \lambda S_n(t).$$

In our competitive model, the parameter λ can be interpreted as permanent price impact, since it quantifies how accumulated inventories $S_n(t)$ affect the price level. A smaller price impact parameter λ implies a deeper or more liquid market. The market tends to be more liquid when there is more disagreement, since traders are more willing to provide liquidity to others. Figure 1 shows that λ decreases in the

degree of disagreement τ_H/τ_L .¹

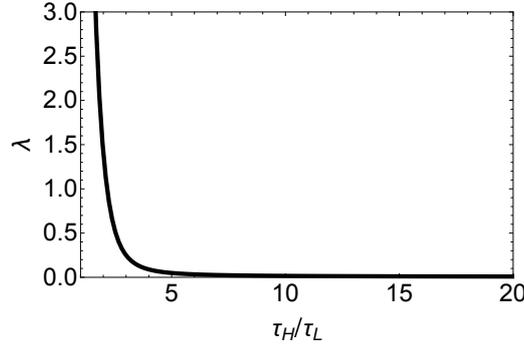


FIGURE 1. λ AGAINST τ_H/τ_L WHILE FIXING τ .

Second, using equation (26), we can calculate expected positions as

$$(36) \quad \begin{aligned} \mathbb{E}\{|S_n(t)|\} &= C_L \mathbb{E}\{|H_n(t) - H_{-n}(t)|\} \\ &= C_L \left(\frac{2}{\pi} \text{Var}\{H_n(t) - H_{-n}(t)\}\right)^{1/2}, \end{aligned}$$

as well as expected position changes related to trading volume $N \mathbb{E}\{|dS_n(t)|\}$.

Figure 2 illustrates how position sizes and trading volume depend on the degree of disagreement.² The size of positions $\mathbb{E}\{|S_n(t)|\}$ increases in the degree of disagreement τ_H/τ_L , as traders tend to hold larger positions in more liquid markets. Trading volume tends to increase with disagreement as well until a reasonably large level of $\tau_H/\tau_L \approx 10$ for the selected parameter values.

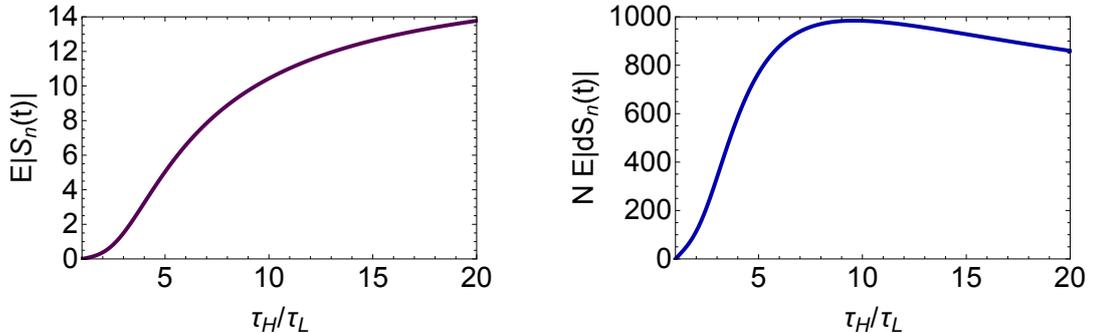


FIGURE 2. $\mathbb{E}\{|S_n(t)|\}$ AND $N \mathbb{E}\{|dS_n(t)|\}$ AGAINST τ_H/τ_L WHILE FIXING τ .

Figure 3 illustrates that both constants C_J and C_G decrease when the degree of disagreement τ_H/τ_L increases while fixing total precision. Disagreement amplifies the dampening effect of beliefs aggregation C_J , since it magnifies the effect of Jensen's inequality. Disagreement also leads to more pronounced price dampening

¹Parameter values are $\tau = 7.4$, $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $\tau_0 = \Omega\sigma_G^2/\sigma_D^2 = 0.0054$, and $N = 100$.

²Parameter values in figures 2 and 3 are the same as those in figure 1.

due to the Keynesian beauty contest C_G , since traders have greater incentives to engage in short-term trading in more liquid markets.

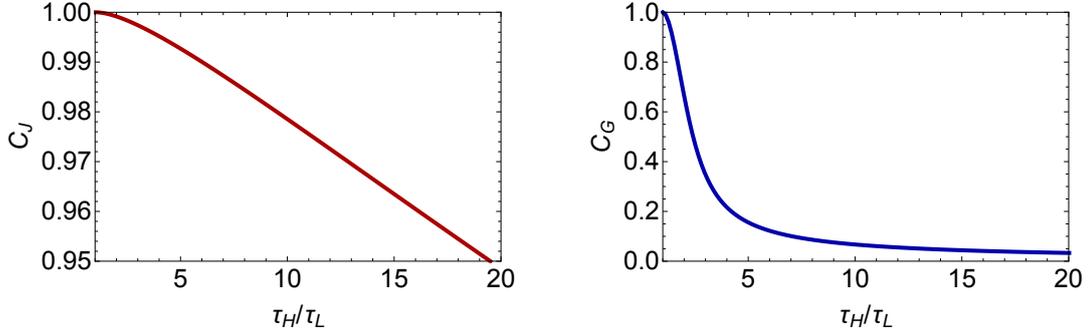


FIGURE 3. C_J AND C_G AGAINST τ_H/τ_L WHILE FIXING τ .

The following proposition describes a limiting case.

PROPOSITION 2: *Assuming $\tau_L = 0$, $\tau_0 \rightarrow 0$, and $N \rightarrow \infty$, the three equations characterizing equilibrium (A-19)–(A-21) have a closed-form solution presented in equations (A-51)–(A-53), implying $\lim_{N \rightarrow \infty} C_G = (r + \alpha_G)/(r + \alpha_G + \tau) < 1$, and $\lim_{N \rightarrow \infty} C_J = 0$.*

The proof is in Appendix A.5. Proposition 2 implies that as the number of traders increases, C_J converges to zero and C_G converges to a constant limit which is less than one. Each trader believes that the other traders observe signals with no information and trade aggressively against one another's perceived mistakes. Even though the market is very liquid, substantial momentum is generated by price dampening.

The following proposition describes how momentum depends on risk aversion.

PROPOSITION 3: *The constants C_J and C_G do not depend on risk aversion A .*

It can be shown that parameters C_J and C_G remain the same when the risk aversion parameter A changes. The proof is in Appendix A.6. The level of risk aversion does not affect the magnitude of momentum. This contrasts our model from other models, such as Vayanos and Woolley (2013), where momentum exists largely due to the limited risk bearing capacity of traders; in their model, larger risk aversion leads to less liquidity and more pronounced momentum. In our paper, momentum is the result of endogenous beliefs dynamics.

4. Beliefs Aggregation and the Representative Agent

In this section, we explain how to think about beliefs aggregation using the construct of the representative agent. We show how to construct beliefs for the representative agent in our model with heterogenous beliefs, and we show that these beliefs are not intuitively simple.

4.1. Beliefs Aggregation

Aggregation is one of the fundamental issues in economics, and economists have extensively studied it from multiple perspectives. We can think of the market as a mechanism which aggregates the public and private *information* of market participants, their *preferences*, their *wealth*, and their *beliefs*.

There is a long literature about aggregation of information. Hayek (1945) writes that designing a rational economic order requires overcoming the key problem that constantly changing information, necessary for making decisions, is dispersed among separate individuals rather than readily available to a central planner. He further suggests that if market participants act in their own interests, the price system is a mechanism that will share information and help to bring about the outcome which “might have been arrived at by one single mind possessing all the information which is in fact dispersed among all the people involved in the process.”

The aggregation is often not straightforward. As discussed in Arrow (1951), for example, it is often difficult to aggregate diverse preferences. One representative agent can aggregate preferences of market participants only under restrictive assumptions, as in our case where traders have exponential utilities. More generally, it is possible to deal with this issue by replacing a single representative agent with a weighted set of agents with different preferences.

Lucas (1978) and Mehra and Prescott (1985) derive insights about economic systems using representative agent models. In these models, all agents act in a manner such that their cumulative actions might as well be the actions of one agent maximizing his expected utility function. The representative-agent assumption allows modelers to focus on economically important properties of the economy—especially when researchers are interested in aggregates such as asset prices—instead of carrying along numerous parameters describing each agent in the models. This idea is closely related to the paradigm of rational expectations equilibrium; an economist with rational expectations is effectively assumed to articulate Muth’s “relevant theory.”

Instead of focusing on aggregation of preferences, we model the market as a mechanism which aggregates the *beliefs* of market participants with different information. Complicated issues related to aggregation of preferences are not relevant for our paper since the risky asset is in zero net supply. Aggregation of information is somewhat simplified, because all relevant information is fully revealed in market prices in our symmetric model, but there are yet non-obvious effects due to different pieces of information being dispersed among traders. Xiong and Yan (2010) and Jouini and Napp (2007) analyze the aggregation of beliefs in settings with no private information but with emphasis on the aggregation of traders’ wealth. In Xiong and Yan (2010), the representative agent’s belief turns out to be the wealth-weighted average belief of traders. In Jouini and Napp (2007), the consensus belief is a weighted-average belief with weights related to risk tolerance of individual traders.

To quantify how the market aggregates beliefs, it is natural to ask whether there

exists a set of parameter values defining a prior distribution which can be attributed to “the market.” If so, then it is also natural to compare this set of parameter values with the empirically correct parameter values used by the economist. If the market’s parameter values are the same as the economist’s parameter values, then the return exhibits no “anomalies;” otherwise there will be predictability.

To implement this idea precisely, we formally introduce the construct of a representative agent. We define the representative agent as a hypothetical trader with the specific beliefs such that he would choose to buy and hold the aggregate endowment (of zero). The assumption of zero net supply implies that the expected return on the risky asset is always equal to the risk-free rate under the beliefs of the representative agent. The beliefs of the representative agent in our model are therefore essentially beliefs about model parameters which are consistent with the *risk-neutral* probabilities introduced in option pricing theory by Merton (1973).

We next show that the representative agent in our model typically disagrees with traders about some parameters that the traders themselves do not disagree about. Researchers thus may need to assign beliefs to the representative agent that are quite different from “consensus” beliefs. Since the link between representative-agent models and their corresponding “first principles” micro-foundations is often not intuitive, reliance on the construct of the representative agent in the literature must not be overly simplified. For example, it may make sense to talk about how traders think and trade, but it may not make sense to talk about how the representative agent thinks and trades.

4.2. The Beliefs of the Representative Agent in the Model

By definition, the beliefs of the representative agent are such that levels and dynamics of his estimates of fundamental value must coincide with market prices. We assume that the representative agent has beliefs which are consistent with the overall information structure described in section 2.1, with the exception that he may assign different values to the parameters α_G , σ_G , and τ_n . As in the motivating examples, we indicate with “breves” (“ $\check{}$ ”) the possibly different parameter values $\check{\alpha}_G$, $\check{\sigma}_G$, and $\check{\tau}_n$ assigned by the representative agent. The representative agent may interpret signals differently from the traders in the model and assign different values $\check{\Omega}$, $\check{\tau}_0$, $\check{\tau}$, and $\check{\tau}_I$ to corresponding parameters (defined below). Since the representative agent agrees with the traders about the values of N , α_D , σ_D , r , ρ , and A , we write these parameter values without breves.

Common-sense intuition suggests that since all traders in the market agree about the value of parameters such as α_G and σ_G (and these values are common knowledge), then the representative agent will have the same beliefs about these parameters as well. Similar intuition also suggests that if traders disagree about the value of other parameters such as τ_n , then the representative agent’s belief about those parameters will be equal to some appropriately weighted average of beliefs of traders. For example, this intuition suggests that the representative agent will assign to each private signal the same precision, equal to some weighted average

of precisions τ_H and τ_L . In contrast to this common-sense intuition, we show that the beliefs of the representative agent may differ from the beliefs suggested by this common-sense intuition. This result is similar to results on beliefs aggregation, where additional degrees of freedom must be introduced to construct a representative agent. Examples include the discount factor in Jouini and Napp (2007) and an adjustment to aggregate wealth in Calvet, Grandmont and Lemaire (2001).

We will first briefly outline information available to the representative agent. The representative agent believes that the unobserved growth rate $G^*(t)$ follows the process

$$(37) \quad dG^*(t) := -\check{\alpha}_G G^*(t) dt + \check{\sigma}_G dB_G(t).$$

The market price aggregates the information content of the divided $D(t)$ and N signals $I_1(t), \dots, I_N(t)$. To keep matters simple, we assume a symmetric information structure in which the representative agent assigns the same precision $\check{\tau}_I$ to all private signals. Otherwise, the representative agent processes information exactly like traders in the model. Each signal $I_n(t)$ then produces a continuous stream of information given by

$$(38) \quad dI_n(t) := \check{\tau}_I^{1/2} \frac{G^*(t)}{\check{\sigma}_G \check{\Omega}^{1/2}} dt + d\check{B}_n(t), \quad n = 1, \dots, N,$$

where $d\check{B}_n(t) = dB_n(t) + (\tau_n/(\sigma_G \Omega^{1/2}) - \check{\tau}_I/(\check{\sigma}_G \check{\Omega}^{1/2}))G^*(t)dt$, and the representative agent believes $dB_G, d\check{B}_1, \dots, d\check{B}_N$ to be independent Brownian motions. To model dividend-information, define $dI_0(t) := (\alpha_D D(t)dt + dD(t))/\sigma_D$, $dB_0 := dB_D$, and

$$(39) \quad \check{\tau}_0 := \frac{\check{\Omega} \check{\sigma}_G^2}{\sigma_D^2}.$$

Then dividend-information can be written

$$(40) \quad dI_0(t) := \check{\tau}_0^{1/2} \frac{G^*(t)}{\check{\sigma}_G \check{\Omega}^{1/2}} dt + dB_0(t).$$

The representative agent believes that the total precision of information is given by

$$(41) \quad \check{\tau} := \check{\tau}_0 + N \check{\tau}_I.$$

The inference problem of the representative agent is also analogous to the inference problem of traders discussed in section 2. Let $\check{E}_t\{\dots\}$ and $\check{\text{Var}}_t\{\dots\}$ denote the representative agent's expectations and variances calculated with respect to his information at time t . The history of each information flow $I_n(t)$ can be summarized by a sufficient statistic $\check{H}_n(t)$ defined as

$$(42) \quad \check{H}_n(t) := \int_{u=-\infty}^t e^{-(\check{\alpha}_G + \check{\tau})(t-u)} dI_n(u), \quad n = 0, 1, \dots, N.$$

Combining private signals and the public signal, define the aggregate sufficient statistic $\check{H}(t)$ as the linear combination of $\check{H}_0(t)$ and $\check{H}_n(t)$, $n = 1, \dots, N$, given by

$$(43) \quad \check{H}(t) = \check{\tau}_0^{1/2} \check{H}_0(t) + \sum_{n=1}^N \check{\tau}_I^{1/2} \check{H}_n(t).$$

Since the representative agent has symmetric beliefs, the statistic $\check{H}(t)$ defined in (43) can be extracted from market prices. Then the representative agent's estimate of the growth rate $\check{G}(t)$ can be written

$$(44) \quad \check{G}(t) := \check{\sigma}_G \check{\Omega}^{1/2} \check{H}(t)$$

with steady-state error variance

$$(45) \quad \check{\Omega} := \check{\text{Var}} \left\{ \frac{G^*(t) - \check{G}(t)}{\check{\sigma}_G} \right\} = \frac{1}{2\check{\alpha}_G + \check{\tau}}.$$

The generalization of the Gordon growth formula under the representative agent's yields the restriction

$$(46) \quad P(t) = \check{\text{E}}_t\{F(t)\} = \frac{D(t)}{r + \alpha_D} + \frac{\check{G}(t)}{(r + \alpha_D)(r + \check{\alpha}_G)},$$

where the price $P(t)$ on the left-hand side is defined in (28). In terms of $\check{H}(t)$, the same restriction can also be written

$$(47) \quad P(t) = \frac{D(t)}{r + \alpha_D} + \frac{\check{\sigma}_G \check{\Omega}^{1/2}}{(r + \alpha_D)(r + \check{\alpha}_G)} \check{H}(t).$$

Since the risky asset is in zero-net supply, the representative agent must have beliefs such that the equilibrium price coincides with his estimate of the fundamental value $P(t) = \check{\text{E}}_t\{F(t)\}$ without any adjustment for a risk premium.

To ensure that the restriction (47) holds, the beliefs for the representative agent must satisfy several conditions. First, signals $H_n(t)$ determining $P(t)$ on the right side of equation (28) must coincide with signals $\check{H}_n(t)$ determining $\check{\text{E}}_t\{F(t)\}$ on the right-hand side of (47). This implies that the decay factor $\check{\alpha}_G + \check{\tau}$ in the definition of $\check{H}_n(t)$ in equation (42) must coincide with the decay factor $\alpha_G + \tau$ in the definition of $H_n(t)$ in equation (23) and results in the first restriction $\check{\alpha}_G + \check{\tau} = \alpha_G + \tau$. Second, the coefficients of the random variables $H_n(t)$ and $\check{H}_n(t)$ must match in equation (47) equating the equilibrium price and the representative agent's estimate of the fundamental value. This leads to two additional restrictions on the model parameters.

The three restrictions imply solutions for the three parameters $\check{\alpha}_G$, $\check{\sigma}_G$, and $\check{\tau}_I$ describing beliefs of the representative agent that are stated in Theorem 2:

THEOREM 2: *Assume the representative agent believes there are N traders whose private signals have the same precisions $\check{\tau}_I$. Then the representative agent's beliefs about the parameters $\check{\alpha}_G$, $\check{\sigma}_G$, and $\check{\tau}_I$ are the following functions of the model parameters:*

$$(48) \quad \check{\tau}_I = \tau_I \frac{C_G (r + \alpha_G + \tau)}{r + \alpha_G + C_G (\tau_0 + N \tau_I)} < \tau_I,$$

$$(49) \quad \check{\alpha}_G = \alpha_G + \frac{r + \alpha_G}{r + \alpha_G + C_G (\tau_0 + N \tau_I)} (\tau - C_G (\tau_0 + N \tau_I)) > \alpha_G,$$

$$(50) \quad \check{\sigma}_G = \sigma_G \left(\frac{C_G (r + \alpha_G + \tau)}{r + \alpha_G + C_G (\tau_0 + N \tau_I)} \left(1 + \frac{\check{\alpha}_G - \alpha_G}{2 \alpha_G + \tau} \right) \right)^{1/2}.$$

Without loss of generality, the representative agent agrees with traders about the values of the parameters α_D , σ_D , r , ρ , and A .

Recall that τ and τ_I are functions of model parameters defined by $\tau := \tau_0 + \tau_H + (N - 1)\tau_L$ from (21) and $\tau_I^{1/2} = (\tau_H^{1/2} + (N - 1)\tau_L^{1/2})/N$ from (30). Also, recall that C_G is a function of model parameters satisfying $0 < C_G \leq 1$. Theorem 2 thus implies that the representative agent's beliefs are non-intuitive functions of model parameters N , r , ρ , A , α_D , σ_D , α_G , σ_G , τ_H , and τ_L . Contrary to common-sense intuition, the representative agent assigns values $\check{\alpha}_G$ and $\check{\sigma}_G$ which differ from the market's consensus values α_G and σ_G .

Aggregation makes the representative agent's beliefs differ from traders' beliefs because beliefs about precisions play two mutually inconsistent roles. On the one hand, these beliefs determine the weights with which the representative agent aggregates incoming information into his current estimate of the growth rate in equation (44); $\check{\tau}_I^{1/2}$ must be defined as a weighted average of square roots of precisions $\tau_H^{1/2}$ and $\tau_L^{1/2}$ to give traders' signals appropriate weights to match the current price levels. On the other hand, these beliefs determine the speed with which the representative agent's signals decay; $\check{\tau}_I$ would have to be defined as a weighted average of τ_H and τ_L (not square roots) in order for signals to have the appropriate decay rate $\alpha_G + \tau$ in the definitions of $H_0(t)$ and $H_n(t)$. In other words, the square roots of precisions determine price volatility while the precisions themselves determine price resilience; both would have to match to construct the representative agent's beliefs in an intuitive manner. Since $\check{\tau}_I$ cannot be defined in both ways at the same time, it is instead necessary for the representative agent's beliefs $\check{\alpha}_G$ and $\check{\sigma}_G$ to differ from traders' beliefs about the value of the corresponding parameters. If we were to assume $\check{\alpha}_G = \alpha_G$ and $\check{\sigma}_G = \sigma_G$, there would be no symmetric beliefs $\check{\tau}_I$ that could simultaneously match both the current price level and its dynamics.

As shown in equations (48), (49), and (50), aggregation leads to several effects. First, the imputed beliefs of the representative agent about the precision of private

signals may be different from the average beliefs of traders in the market; the price dampening effects from beliefs aggregation and the Keynesian beauty contest lower the weight on private signals. Second, the mean-reversion parameter $\check{\alpha}_G$ of the representative agent must be larger than the mean-reversion parameter α_G of the traders; this persistency dampening is necessary to make the decay rate of signals consistent with a lower private signal precision. Third, the volatility parameter $\check{\sigma}_G$ of the representative agent can be either higher or lower than the traders' dividend growth volatility σ_G , depending on particular parameters.

While, as emphasized by Samuelson (1965), one trader's valuation process has a martingale property, it is well-known that the average of martingales is not necessarily a martingale when the martingales are not independent. In our model, the market price, which averages the martingale expectations of traders, does not have a martingale property. For the price to have a martingale property, beliefs about the parameters governing the filtration must be changed in a non-intuitive way.

Our discussion shows that it is misleading—or even incorrect—to infer that parameters describing the beliefs of the representative agent are simple arithmetic averages of parameters describing beliefs of traders in a market. The interactions among individual traders with heterogeneous information in a dynamic model can make their average beliefs quite different from the beliefs of the representative agent. Guessing these beliefs without solving the micro-founded model would be impossible. A detailed modeling of the interactions among individual traders thus might still be necessary to generate further economic insights in addition to the representative agent models. The effects discussed above from beliefs aggregation can significantly affect the dynamics of equilibrium prices.

5. Return Dynamics and Return Predictability

Designing empirical tests is difficult for markets where traders have different beliefs. Next, we present the endogenously derived structural model for returns and discuss their time-series properties in the context of our model. This exercise may provide some guidance for empirical research on return predictability in markets with heterogeneous beliefs, related to Greenwood and Shleifer (2014) and Buraschi, Piatti and Whelan (2016) among others.

In markets with heterogeneous beliefs, it is important to make a distinction between parameter values defined by traders' beliefs and empirically correct parameter values. Since each trader believes his own private signal is more precise than other traders believe it to be, these calculations cannot all be empirically correct. In a symmetric model, in which the empirically correct precision of all traders' signals are the same, none of the individual traders' beliefs are correct. Empirically correct model outcomes—such as a trader's expected profits and expected paper-trading returns on the risky asset—depend on *both* the possibly incorrect parameters used by the traders *and* the empirically correct parameters used by the economist.

For simplicity of discussion, we assume that an *economist* has perfectly accurate point estimates of model parameters. This is intuitively consistent with the idea that the economist has access to an infinite amount of data to estimate model parameters consistently, assuming the values of the parameters are statistically identified. More generally, we could think of the economist as a statistician who estimates model parameters with some degree of statistical error, but this would take us beyond the scope of this paper. We also assume that the empirically correct precision of each trader's signal is the same.

In the specific case when the beliefs of the economist correspond to the beliefs of the representative agent, the expected return is equal to the risk-free rate. Otherwise, the expected return is different from the risk-free rate, and the expected return is a complicated function of the entire history of dividends and prices.

5.1. The Economist's Inference Problem

We start by introducing empirically correct beliefs about model parameters. As in the motivating examples, we use “hats” to distinguish the beliefs of the economist from the beliefs of the traders.

The economist with empirically correct beliefs assigns precision $\hat{\tau}_0$ to public information and $\hat{\tau}_I$ to each private signal process. From the economist's perspective, the total precision is $\hat{\tau} = \hat{\tau}_0 + N \hat{\tau}_I$. From the perspective of each trader (but not the representative agent), the total precision is $\tau = \tau_0 + \tau_H + (N - 1) \tau_L$. In general, these precisions are different ($\hat{\tau} \neq \tau$).

Except for beliefs about the parameters $\hat{\alpha}_G$, $\hat{\sigma}_G$, and $\hat{\tau}_I$, we assume that the economist has the same beliefs about parameter values as the traders. In particular, we assume that the economist and traders agree about the parameters α_D and σ_D . Note that the value of σ_D can be inferred with perfect accuracy from observing the dividend process $D(t)$ continuously.

By placing “hats” over the variables in equations (37), (38), (39), (41), (45), (42), (43), and (44) above, we obtain definitions of $\hat{\Omega}$, $\hat{\tau}_0$, $\hat{\tau}$, and $\hat{H}_n(t)$ for $n = 0, 1, \dots, N$, which are consistent with the economist's expectation operator $\hat{E}_t\{\dots\}$. The calculations are analogous to those derived under the beliefs of the traders and the representative agent. We briefly summarize these definitions here:

$$(51) \quad dG^*(t) = -\hat{\alpha}_G G^*(t) dt + \hat{\sigma}_G dB_G(t),$$

$$(52) \quad \hat{\tau}_0 := \frac{\hat{\Omega} \hat{\sigma}_G^2}{\sigma_D^2},$$

$$(53) \quad \hat{\tau} = \hat{\tau}_0 + N \hat{\tau}_I,$$

$$(54) \quad \hat{\Omega} := \hat{\text{Var}} \left\{ \frac{G^*(t) - \hat{G}(t)}{\hat{\sigma}_G} \right\} = \frac{1}{2 \hat{\alpha}_G + \hat{\tau}},$$

$$(55) \quad dI_0(t) := \hat{\tau}_0^{1/2} \frac{G^*(t)}{\hat{\sigma}_G \hat{\Omega}^{1/2}} dt + dB_0(t),$$

$$(56) \quad dI_n(t) := \hat{\tau}_I^{1/2} \frac{G^*(t)}{\hat{\sigma}_G \hat{\Omega}^{1/2}} dt + d\hat{B}_n(t), \quad n = 1, \dots, N,$$

$$(57) \quad d\hat{B}_n(t) = dB_n(t) + \left(\frac{\tau_n}{\sigma_G \Omega^{1/2}} - \frac{\hat{\tau}_n}{\hat{\sigma}_G \hat{\Omega}^{1/2}} \right) G^*(t) dt,$$

$$(58) \quad \hat{H}_n(t) := \int_{u=-\infty}^t e^{-(\hat{\alpha}_G + \hat{\tau})(t-u)} dI_n(u), \quad n = 0, 1, \dots, N,$$

$$(59) \quad \hat{H}(t) = \hat{\tau}_0^{1/2} \hat{H}_0(t) + \sum_{n=1}^N \hat{\tau}_I^{1/2} \hat{H}_n(t),$$

$$(60) \quad \hat{G}(t) := \hat{E}\{G^*(t)\} = \hat{\sigma}_G \hat{\Omega}^{1/2} \hat{H}(t).$$

As can be seen from equations (23) and (58), both the traders and the economist construct their sufficient statistics $\hat{H}_n(t)$ and $H_n(t)$ as linear combinations of increments in information flow, with weights decaying exponentially over time. While the decay rate used by the representative agent is by definition the same as the decay rate used by the traders, the correct decay rate used by the economist may be different. Therefore, in general we have

$$(61) \quad \hat{\alpha}_G + \hat{\tau} \neq \alpha_G + \tau = \check{\alpha}_G + \check{\tau}.$$

It can be shown that the sufficient statistics $\hat{H}_n(t)$ and $H_n(t)$, $n = 0, 1, \dots, N$, relate to each other as follows,

$$(62) \quad \hat{H}_n(t) = H_n(t) + (\alpha_G + \tau - \hat{\alpha}_G - \hat{\tau}) \int_{u=-\infty}^t e^{-(\hat{\alpha}_G + \hat{\tau})(t-u)} H_n(u) du.$$

If the economist agrees with the traders about both the mean-reversion rate ($\alpha_G = \hat{\alpha}_G$) and the total precision of the signals ($\tau = \hat{\tau}$), then the market's and economist's statistics coincide, yielding $\hat{H}_n(t) = H_n(t)$. If the economist disagrees with traders about how quickly information decays, then the sufficient statistics $\hat{H}_n(t)$ and $H_n(t)$ are different, and the relationship between the two sufficient statistics depends on the entire history of information flow. For example, the economist may assign higher weights to the information from the distant past if he believes that dividends are more persistent or signals are less precise than traders believe. In this case,

we have $\alpha_G + \tau > \hat{\alpha}_G + \hat{\tau}$, and equation (62) shows how to obtain the economist's sufficient statistic $\hat{H}_n(t)$ for trader n 's signal as a function of the infinite history of a trader n 's sufficient statistic $H_n(t)$.

We will be mostly interested in the aggregate of the economist's statistics $\hat{H}(t)$ defined in (59). Since the histories of $H_0(t)$ and $H(t)$ can be recovered from the histories of dividends and prices, equation (62) implies that the economist can recover his statistics $\hat{H}(t)$ from dividends and prices as well. We will show next that the expected return from the perspective of the economist is not the risk-free rate, *even when traders are correct on average*, but rather has a specific closed form which depends on current and past prices and dividends.

5.2. Predictability of the Instantaneous Return

From the perspective of the economist, the equilibrium return process has a linear structure that depends on the economist's sufficient statistics $\hat{H}(t)$ and the market's sufficient statistics $H(t)$. Using equation (28), which expresses the market price $P(t)$ as a function of the dividend $D(t)$ and the market's sufficient statistic $H(t)$, we can write an equation for $dP(t)$, plug in $dH_n(t)$ using equation (23), and plug in the economist's beliefs about the dynamics of $dI_n(t)$ from equation (56) and the economist's estimate $\hat{G}(t)$ from equation (60). This yields an equation for the instantaneous return given by

$$(63) \quad dP(t) + D(t) dt = r P(t) dt + (b \hat{H}(t) - a H(t)) dt + d\hat{B}_r(t),$$

where the coefficients a and b are defined as

$$(64) \quad a := \frac{\sigma_G C_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} (\alpha_G + r + \tau),$$

$$(65) \quad b := \frac{\hat{\sigma}_G \hat{\Omega}^{1/2}}{r + \alpha_D} + \frac{\sigma_G C_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \left(\tau_0^{1/2} \hat{\tau}_0^{1/2} + \tau_I^{1/2} N \hat{\tau}_I^{1/2} \right),$$

and the instantaneous variance of the return is given by

$$(66) \quad \hat{\text{Var}} \left\{ \frac{d\hat{B}_r(t)}{dt^{1/2}} \right\} = \left(\frac{\sigma_D}{r + \alpha_D} + \frac{\sigma_G \Omega^{1/2} C_G \tau_0^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \right)^2 + \frac{(\sigma_G \Omega^{1/2} C_G)^2 N \tau_I}{(r + \alpha_D)^2 (r + \alpha_G)^2}.$$

Recall that $\hat{\mathbb{E}}_t\{\dots\}$ and $\hat{\text{Var}}_t\{\dots\}$ denote expectation and variance operators based on the empirically correct beliefs of the economist calculated using information available at time t . Since $d\hat{B}_r(t)$ is a martingale with respect to $H(t)$ and $\hat{H}(t)$, the expected return, conditional on all public and private information, is given by

$$(67) \quad \hat{\mathbb{E}}_t \left\{ \frac{dP(t) + D(t) dt}{dt} \right\} = r P(t) + b \hat{H}(t) - a H(t).$$

The expected return is a linear combination of the average sufficient statistic $H(t)$ of traders and the sufficient statistic $\hat{H}(t)$ of the economist. This formula implies a complicated path-dependent and auto-correlated return process. It generalizes to a dynamic environment equation (4) in the first motivating example and equation (8) in the second motivating example.

The expected return is equal to the risk-free rate $r P(t)$ if and only if the beliefs of the representative agent happen to coincide with the empirically correct beliefs of the economist. The model is consistent with the weak rational expectations hypothesis only for the very particular sets of parameters and beliefs given in Theorem 2. Theorem 2 suggests that this is unlikely, because the beliefs of the representative agent are usually different from the average beliefs of traders. Equation (67) shows that the expected return is time varying, depending in a complicated manner on the entire history of past signals. Unless market participants' beliefs about parameters are incorrect in a very specific manner, the economist will see profit opportunities and believe expected-return dynamics to depend on several state variables.

The analysis reveals that the endogenous time-series momentum due to beliefs aggregation and the Keynesian beauty contest continues to influence return dynamics. Return dynamics are functions of the dynamically changing statistics used both by traders in the market and by the economist. To develop some intuition, consider a special case in which both traders and the economist agree on the total precision of the information flow ($\hat{\tau} = \tau$) and the parameters describing the univariate dynamics of the growth rate ($\hat{\alpha}_G = \alpha_G$ and $\hat{\sigma}_G = \sigma_G$); this is one way to formalize the intuition that traders' beliefs are "correct on average." Then, both $\hat{H}_n(t)$ and $H_n(t)$ mean-revert at the same rate, implying $\hat{H}_n(t) = H_n(t)$, but $\hat{H}(t) \neq H(t)$ because the weights $\hat{\tau}_I^{1/2}$ and $\tau_I^{1/2}$ in their definitions are different. In terms of $H_0(t)$ and $H_n(t)$, the risk premium can be written

$$(68) \quad b \hat{H}(t) - a H(t) = \frac{\sigma_G \Omega^{1/2}}{r + \alpha_D} \left((1 - C_G) \tau_0^{1/2} - \frac{C_G N \hat{\tau}_I^{1/2} \tau_0^{1/2}}{r + \alpha_G} (\hat{\tau}_I^{1/2} - \tau_I^{1/2}) \right) H_0(t) \\ + \frac{\sigma_G \Omega^{1/2}}{r + \alpha_D} \left((1 - C_G) \tau_I^{1/2} + \frac{r + \alpha_G + C_G \tau_0}{r + \alpha_G} (\hat{\tau}_I^{1/2} - \tau_I^{1/2}) \right) \sum_{n=1}^N H_n(t).$$

It can be shown that the coefficient on $\sum_{n=1}^N H_n(t)$ in this expression is always positive. Indeed, its first term with $1 - C_G > 0$ results from the price dampening effect of the Keynesian beauty contest, and its second term with $\hat{\tau}_I^{1/2} - \tau_I^{1/2} > 0$ results from the price dampening effect of beliefs aggregation. Thus, there will be the momentum in price dynamics even when traders and the economist agree on the total precision of the information flow and other parameters of the model.

5.3. Implications for VAR Frameworks

We can now write the return dynamics (63) in a more intuitive and familiar form. Since the price $P(t)$ is a linear combination of $D(t)$ and $H(t)$ from equation (28) and since $\hat{H}(t)$ can be recovered from the history of $H(t)$ using equation (62), both $H(t)$ and $\hat{H}(t)$ can be recovered from the history of prices $P(t)$ and dividends $D(t)$. This allows us to show that the return process, conditional on all public and private information, depends in a specific manner on the history of publicly observable dividends $D(t)$ and prices $P(t)$:

THEOREM 3: *The equilibrium return dynamics can be expressed as a linear combination of past publicly observable dividends $D(t)$ and prices $P(t)$,*

$$(69) \quad \begin{aligned} dP(t) + D(t) dt &= r P(t) dt + \alpha_1 \left(P(t) - \frac{D(t)}{r + \alpha_D} \right) dt \\ &+ \alpha_2 \left(\int_{u=-\infty}^t \left(P(u) - \frac{D(u)}{r + \alpha_D} \right) e^{-(\hat{\alpha}_G + \hat{\tau})(t-u)} du \right) dt \\ &- \alpha_3 \left(\int_{u=-\infty}^t e^{-(\hat{\alpha}_G + \hat{\tau})(t-u)} dI_0(u) \right) dt + d\hat{B}_r(t), \end{aligned}$$

where the constants α_1 , α_2 , and α_3 are defined in equations (A-68), (A-69), and (A-70) in Appendix A.8 and $d\hat{B}_r(t)$, defined in equation (A-63), is a martingale increment with respect to all information at time t .

Equation (69) has a simple intuition. First, investors obtain the unconditional expected return of $r P(t)$. Second, investors obtain a conditional excess return proportional to the deviation of the *current* price $P(t)$ from the unconditional valuation $D(t)/(r + \alpha_D)$. Third, investors obtain a conditional excess return proportional to the *past* deviations of prices from the unconditional valuation and the *past* dividends surprises dI_0 ; the importance of each past component decays exponentially at rate $\hat{\alpha}_G + \hat{\tau}$.

In general, our structural model implies that whether the coefficients α_1 , α_2 , and α_3 are positive or negative depends on how the beliefs of traders about parameter values deviate from the empirically correct values of these parameters.

For some particular cases, the expected return depends only on the current dividend-price relationship $P(t) - D(t)/(r + \alpha_D)$ and not its past values. By definition, the expected return equals the risk-free rate when the economist's correct parameter values are the same as those of the representative agent.

COROLLARY 1: *Assume $\alpha_G + \tau = \hat{\alpha}_G + \hat{\tau}$. Then we have $\alpha_2 = 0$.*

This result states that if the information decay rate $\alpha_G + \tau$ used by traders is correct, then the expected return depends only on the current value $P(t) - D(t)/(r + \alpha_D)$ but not its past values; the expected return also depends on past dividend innovations through the α_3 -term. Since the representative agent also uses

the same decay rate as the market, $\check{\alpha}_G + \check{\tau} = \hat{\alpha}_G + \hat{\tau}$, the representative agent's decay rate is in this case also empirically correct.

COROLLARY 2: *Assume $\hat{\alpha}_G = \alpha_G$, $\hat{\sigma}_G = \sigma_G$, and $\hat{\tau} = \tau$. Then we have $\alpha_1 > 0$, $\alpha_2 = 0$, and $\alpha_3 > 0$.*

The inequalities in this corollary can be proved by noticing that $\hat{\Omega} = \Omega$ and $\hat{\tau}_0 = \tau_0$ imply $\hat{\tau}_I > \tau_I$ using Jensen's inequality.

This result describes a more restrictive case in which the market is assumed to use the empirically correct values of $\hat{\alpha}_G$ and $\hat{\sigma}_G$. Since $\hat{\tau} = \tau$ implies $N \hat{\tau}_I = \tau_H + (N - 1)\tau_L$, the market's beliefs about the precisions of signals are "correct on average." In this sense, traders are relatively overconfident but not absolutely overconfident or under-confident. The result $\alpha_2 = 0$ says that the expected return does not depend on past values of $P(t) - D(t)/(r + \alpha_D)$. The result $\alpha_1 > 0$ has the interpretation that prices under-react to private information. This implies that a price above (below) its unconditional level predicts a high (low) return in the short run.

The relationship between prices and dividends has been extensively tested in the literature. Our CARA-normal approach makes it convenient to measure the return in dollars per share per unit of time, not as a percentage of the asset's value. It also makes it convenient to measure the relationship between prices and dividends as an arithmetic difference, not a ratio. In our model, exponential utility and normal random variables are simplifications. A model more amenable to empirical estimation should have constant relative risk aversion rather than constant absolute risk aversion and log-normal random variables rather than normally distributed random variables. To apply results from our CARA-normal setting to the empirical literature, it is useful to think of the difference between the price and unconditional valuation based on dividends alone, $P(t) - D(t)/(r + \alpha_D)$, as expressing a "dividend-price relationship" analogous to a dividend-price ratio in the empirical literature.

Similar to the auto-regressive equation (69) for the return, the auto-regressive equations for price-dividend differences $P(t) - D(t)/(r + \alpha_D)$ and dividends $D(t)$ can be derived as well. As reviewed in Cochrane (2008), the VAR system comprised of these three equations then will be analogous to the VAR framework which is widely used in the empirical studies of return predictability. Our structural model provides both theoretical underpinning for these studies and guidance for new research directions.

For example, our model suggests that when $\alpha_G + \tau \neq \hat{\alpha}_G + \hat{\tau}$, it is the entire history of the dividend-to-price ratios and dividends—not only their current values—that should be included as explanatory variables in return-forecasting regressions in order to capture all information relevant for predicting the return. Thus, it may be warranted to consider more carefully VAR models with multiple lags, like Campbell and Shiller (1988), rather than a VAR model with one lag as is typical in more recent literature.

Our structural economic model also implies specific non-linear restrictions on the coefficients governing the relationship between the histories of dividends, prices, and expected returns, linking them to the deep parameters of the model. The coefficients on the past price-dividend relationship are predicted to decay exponentially over time. Data on quantities traded can also be added to the VAR system. This allows the model to connect returns predictability with the holding horizons and trading volumes of traders. Testing these restrictions is an interesting issue for future research.

5.4. Predictability of the Holding-Period Return

We can also calculate the holding-period return under the empirically correct beliefs of an economist who assigns precision $\hat{\tau}_0$ to the public signal and precision $\hat{\tau}_I$ to each private signal.

Let $R(t, t + T)$ denote the cumulative un-discounted holding-period mark-to-market cash flow per share on a fully levered investment in the risky asset from time t to time $t + T$:

$$(70) \quad R(t, t + T) = \int_{u=t}^{t+T} \left(dP(u) + D(u) du - r P(u) du \right).$$

From equation (67), we obtain

$$(71) \quad R(t, t + T) = \int_{u=t}^{t+T} \left(b \hat{H}(u) - a H(u) \right) du + \int_{u=t}^{t+T} d\hat{B}_r(u).$$

The following theorem formally describes a structural model for holding-period returns.

THEOREM 4: *The holding-period return $R(t, t + T)$ can be represented as*

$$(72) \quad R(t, t + T) = \beta_2(T) \hat{H}(t) - \beta_1(T) H(t) + \bar{B}(t, t + T),$$

where time-varying coefficients $\beta_1(T) > 0$ and $\beta_2(T) > 0$ are defined by

$$(73) \quad \beta_1(T) := \frac{a}{\alpha_G + \tau} \left(1 - e^{-(\alpha_G + \tau)T} \right),$$

$$(74) \quad \beta_2(T) := b \frac{1 - e^{-\hat{\alpha}_G T}}{\hat{\alpha}_G} - a \frac{\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2}}{\hat{\alpha}_G (\alpha_G + \tau)} \left(1 + \frac{\hat{\alpha}_G e^{-(\alpha_G + \tau)T} - (\alpha_G + \tau) e^{-\hat{\alpha}_G T}}{\tau + \alpha_G - \hat{\alpha}_G} \right),$$

and $\bar{B}(t, t + T)$ is a martingale increment defined in equation (A-76) in Appendix A.9. The constants a and b are as defined in equations (64) and (65).

The expected holding-period returns depend both on the traders' interpretations of information aggregated into statistic $H(t)$ and the economist's interpretation of

information aggregated into statistic $\hat{H}(t)$. The average beliefs of traders $H(t)$ anchor current prices, whereas the empirically correct beliefs of the economist $\hat{H}(t)$ anchor fundamentals and therefore long-run price levels. For example, the theorem implies that the expected holding-period return is positive if and only if the economist's signal $\hat{H}(t)$ is greater than $\beta_1(T)/\beta_2(T) H(t)$. We later illustrate some properties of this relationship.

In a special case when the economist and the traders agree on the total precision of information flow and other parameters of the model— $\hat{\tau} = \tau$, $\hat{\alpha}_G = \alpha_G$, and $\hat{\sigma}_G = \sigma_G$ —we obtain the following proposition:

PROPOSITION 4: *If the economist and the traders agree on the total precision of information flow and other parameters of the model— $\hat{\tau} = \tau$, $\hat{\alpha}_G = \alpha_G$, and $\hat{\sigma}_G = \sigma_G$,—then the expected return can be written*

(75)

$$E_t\{R(t, t+T)\} = (\beta_2(T) - \beta_1(T)) \tau_0^{1/2} H_0(t) + \left(\beta_2(T) \hat{\tau}_I^{1/2} - \beta_1(T) \tau_I^{1/2} \right) \sum_{n=1}^N H_n(t).$$

The coefficient on $H_0(t)$ monotonically increases in the horizon T for $T \geq T_1$, where T_1 is defined in (A-100) in Appendix A.12. The coefficient on $\sum_{n=1}^N H_n(t)$ is positive and monotonically increases in the horizon T .

The expected holding-period returns tend to exhibit momentum due to the price dampening effects from both beliefs aggregation and the Keynesian beauty contest. This makes the coefficient $\beta_2(T) \hat{\tau}_I^{1/2} - \beta_1(T) \tau_I^{1/2}$ on $\sum_{n=1}^N H_n(t)$ positive. In Appendix A.12, we show that the coefficient can be decomposed to two terms, where the first term with $1 - C_G > 0$ results from the price dampening effect of the Keynesian beauty contest and the second term with $\hat{\tau}_I^{1/2} - \tau_I^{1/2} > 0$ results from the price dampening effect of beliefs aggregation.

In a general case, a wide range of return patterns may arise in the equilibrium, when the economist and traders disagree about parameter values such as $\hat{\alpha}_G$, $\hat{\sigma}_G$, and $\hat{\tau}$. Some of these patterns are consistent with the empirical evidence of momentum in the short run and mean-reversion in the long run. We illustrate these patterns analytically and provide several numerical examples in the next section.

5.5. Autocorrelation of the Holding-Period Return

To develop the intuition that the holding-period return depends on parameter values and the current values of the sufficient statistics $H(t)$ and $\hat{H}(t)$, we analyze and provide examples for three specific combinations of different parameter values (cases A, B, and C) and several combinations of $H(t)$ and $\hat{H}(t)$. The values of $H(t)$ and $\hat{H}(t)$ are jointly normally distributed; their covariance matrix is derived in Appendix A.10. Since the patterns of the expected holding-period return for negative $H(t)$ are symmetric to those for positive $H(t)$, we only present cases with positive $H(t)$.

Case A. Figure 4 plots the expected holding-period return for different horizons T and for different levels of disagreement (τ_H/τ_L) when both the economist and traders agree on the total precision of information flow ($\hat{\tau} = \tau$) and other parameters of the model ($\hat{\alpha}_G = \alpha_G$ and $\hat{\sigma}_G = \sigma_G$).³ In this case, the dynamics of $\hat{H}_n(t)$ coincides with the dynamics of $H_n(t)$. Proposition 4 implies that we tend to have momentum in returns since the coefficient of $\sum_{n=1}^N H_n(t)$ is positive and monotonically increases in horizon T . The left panel of figure 4 plots the expected holding-period return when $H(t) > 0$ is +1 standard deviation from its unconditional mean. The right panel of figure 4 plots the expected holding-period return when $H(t) > 0$ is +2 standard deviations from its unconditional mean.

Figure 4 exhibits monotonically increasing curves. Since we consider positive $H(t)$, which loosely speaking corresponds to positive past returns, the upward sloping curves imply momentum in return dynamics. Momentum occurs due to the price dampening effect from both beliefs aggregation and the Keynesian beauty contest as explained before. The magnitude of momentum increases with the level of disagreement (τ_H/τ_L).⁴ Since market tends to be more liquid with more disagreement, our model implies that momentum tends to be more pronounced in more liquid markets.

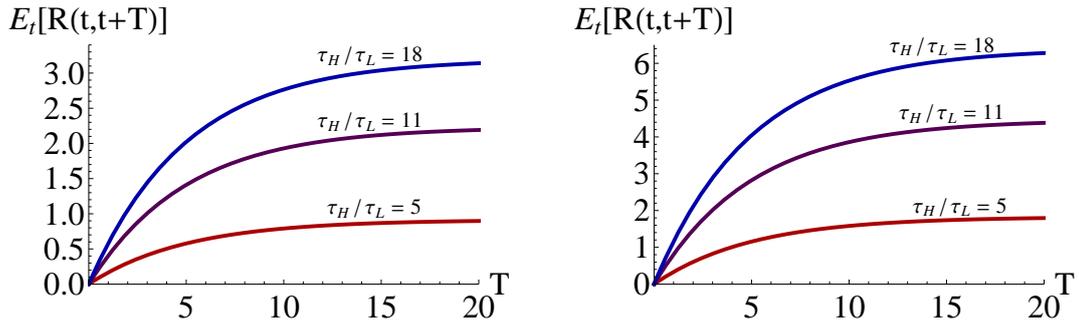


FIGURE 4. THE EXPECTED HOLDING-PERIOD RETURN $R(t, t + T)$ FOR DIFFERENT HORIZONS T FOR THE CASE WHEN $\tau = \hat{\tau}$, $\hat{\alpha}_G = \alpha_G$, AND $\hat{\sigma}_G = \sigma_G$. THE VALUES $H_0(t) = 0$ AND $\hat{H}(t) = \hat{\tau}_I^{1/2}/\tau_I^{1/2} H(t) > 0$ ARE FIXED. THE VALUE OF $H(t)$ IS +1 STANDARD DEVIATION ON THE LEFT SUBPLOT AND +2 STANDARD DEVIATIONS $H(t)$ ON THE RIGHT SUBPLOT.

We also compute the auto-covariances $\text{Cov}\{R(t - T_l, t), R(t, t + T_f)\}$ and auto-correlations $\text{Corr}\{R(t - T_l, t), R(t, t + T_f)\}$ of the cumulative return for different lags T_l and leads T_f . The positive auto-covariance implies that the conditional expected return $E\{R(t, t + T_f) \mid R(t - T_l, t)\}$ is increasing in $R(t - T_l, t)$, thus indicating

³The parameters are $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.2$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $\hat{\tau}_0 = \tau_0 = 0.016$, $\tau_L = 0.019$, $\tau_H = 0.1$, and $N = 100$.

⁴We use $\tau_L = 0.018$, $\tau_H = 0.2$ and $\tau_L = 0.017$, $\tau_H = 0.3$ for the other two disagreement levels in figure 4.

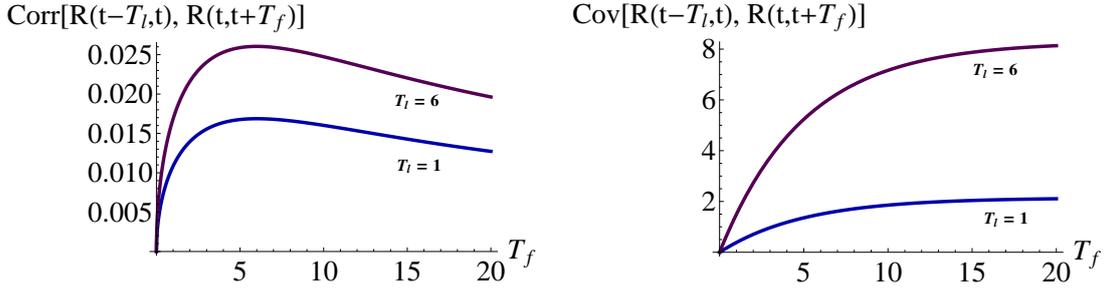


FIGURE 5. THE COVARIANCE $\text{Cov}\{R(t - T_l, t), R(t, t + T_f)\}$ AND CORRELATION $\text{Corr}\{R(t - T_l, t), R(t, t + T_f)\}$ FOR DIFFERENT LEADS T_f AND LAGS FIXED AT $T_l = 1, 6$, WITH $\tau = \hat{\tau}$, $\hat{\alpha}_G = \alpha_G$, AND $\hat{\sigma}_G = \sigma_G$.

momentum in the sense that higher returns in the past tend to be followed by higher returns in the future. The details are presented in equations (A-82), (A-98), and (A-99) in Appendix A.11. Figure 5 plots these covariances and correlations for different leads $T_f = 1, \dots, 20$, and lags fixed at $T_l = 1, 6$. All correlations and covariances are positive, indicating time-series momentum. As expected, when the lead horizon T_f increases, the holding-period return converges asymptotically to a constant, covariances converge to some long-run positive level, and correlations converge to zero due to the quickly increasing variance of holding-period returns.

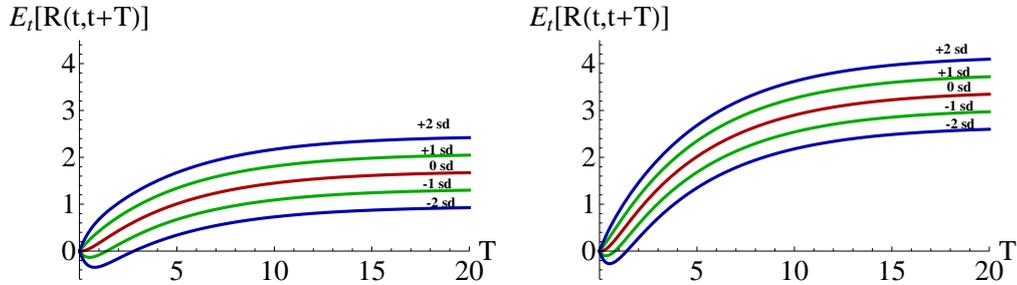


FIGURE 6. THE EXPECTED HOLDING-PERIOD RETURN $R(t, t + T)$ FOR DIFFERENT HORIZONS T FOR THE CASE WHEN $\tau > \hat{\tau}$, $\hat{\alpha}_G = \alpha_G$, AND $\hat{\sigma}_G = \sigma_G$. THE VALUE OF $H(t) > 0$ IS +1 STANDARD DEVIATION ON THE LEFT SUBPLOT AND +2 STANDARD DEVIATIONS ON THE RIGHT SUBPLOT. THE VALUES OF $\hat{H}(t)$ CONDITIONAL ON $H(t)$ ARE +2, +1, 0, -1, -2 STANDARD DEVIATIONS.

Case B. Figure 6 illustrates the case when traders and the economist agree on the parameters of the model ($\hat{\alpha}_G = \alpha_G$ and $\hat{\sigma}_G = \sigma_G$) but disagree about the total

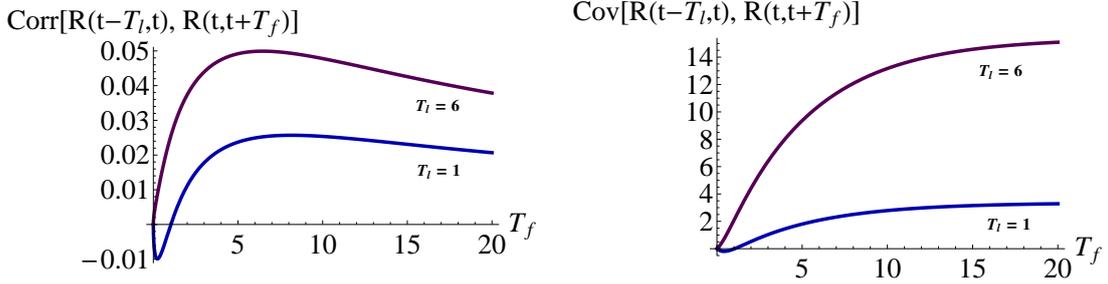


FIGURE 7. THE COVARIANCE $\text{Cov}\{R(t - T_l, t), R(t, t + T_f)\}$ AND CORRELATION $\text{Corr}\{R(t - T_l, t), R(t, t + T_f)\}$ FOR DIFFERENT LEADS T_f AND LAGS FIXED AT $T_l = 1, 6$, WITH $\tau > \hat{\tau}$, $\hat{\alpha}_G = \alpha_G$, AND $\hat{\sigma}_G = \sigma_G$.

precision of the information flow. Traders are absolutely overconfident ($\tau > \hat{\tau}$).⁵ The left subplot depicts the expected holding-period return when $H(t) > 0$ is +1 standard deviation from its unconditional mean. The right subplot depicts the expected holding-period return when $H(t) > 0$ is +2 standard deviations from its unconditional mean. For both cases, $-2, -1, 0, +1, +2$ standard deviations for $\hat{H}(t)$ conditional on $H(t)$ are plotted. Proposition 5 in Appendix A.12 analytically proves that there are only two patterns of the expected holding period return in this case: (1) only momentum, or (2) short-run reversal followed by long-run momentum. Proposition 5 gives specific conditions for each pattern to occur.

As illustrated in figure 6, the momentum effect continues to dominate return dynamics for most situations. Most of the curves are upward sloping, except for several situations when the current signal of the economist $\hat{H}(t)$ is very low relative to the signal of traders $H(t)$ and the return dynamics exhibit a slight mean-reversion in the short run before the momentum effect starts dominating in the long run. Figure 7 also depicts the covariance and correlation of the cumulative return. Strong momentum still makes most of the correlations and covariances positive. For a few sets of parameters, some correlations and covariances have negative values at very short horizons, consistent with figure 6.

Case C. Figure 8 illustrates a more general case when the economist and traders disagree about both the total precision of the information flow and the parameters of the model.⁶ The left subplot depicts the expected holding-period return when $H(t) > 0$ is +1 standard deviation from its unconditional mean. The right subplot depicts the expected holding-period return when $H(t) > 0$ is +2 standard deviations from its unconditional mean. In each subplot, $\hat{H}(t)$ takes values $-2, -1, 0, +1, +2$ standard deviations away from its conditional mean. From

⁵The parameters are the same as before, except $\hat{\tau} = 1.6$ and $\tau = 2.02$.

⁶We assume $\tau = 2.02 > \hat{\tau} = 1.6$, $\hat{\alpha}_G = 1 > \alpha_G = 0.2$, and $\hat{\sigma}_G = 0.5 > \sigma_G = 0.1$.

proposition 5 in Appendix A.12, the implied return patterns are empirically realistic in the sense that the return exhibits momentum in the short run and mean-reversion in the long run if $H(t)/\nu_1 \leq \hat{H}(t) < H(t)/\nu_2$ and traders believe that the growth rate is highly volatile (high values of σ_G) or highly persistent (low values of α_G), where ν_1 and ν_2 are defined in (A-102) and (A-103). This is consistent with our second motivating example, where there is mean reversion in returns if traders believe that the growth rate is more persistent than it actually is. Figure 9 depicts correlations and covariances, the signs of which are consistent with these patterns.

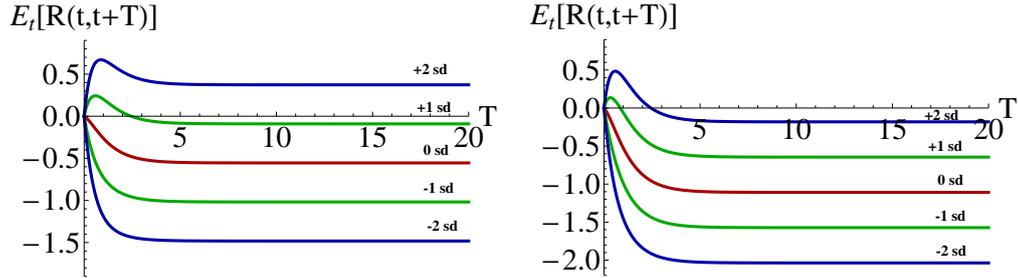


FIGURE 8. THE EXPECTED HOLDING-PERIOD RETURN $R(t, t+T)$ FOR DIFFERENT HORIZONS T FOR THE CASE $\tau > \hat{\tau}$, $\hat{\alpha}_G \neq \alpha_G$, AND $\hat{\sigma}_G \neq \sigma_G$. THE VALUE OF $H(t) > 0$ IS +1 STANDARD DEVIATION ON THE LEFT SUBPLOT AND +2 STANDARD DEVIATIONS ON THE RIGHT SUBPLOT. THE VALUES OF $\hat{H}(t)$ CONDITIONAL ON $H(t)$ ARE +2, +1, 0, -1, -2 STANDARD DEVIATIONS.

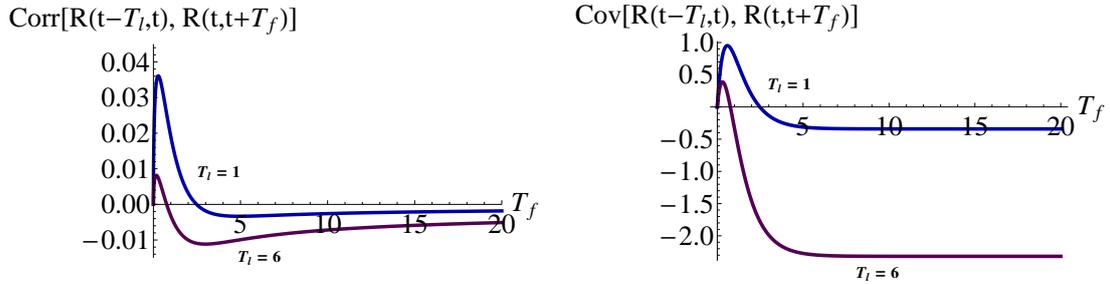


FIGURE 9. THE COVARIANCE $\text{Cov}\{R(t - T_i, t), R(t, t + T_f)\}$ AND CORRELATION $\text{Corr}\{R(t - T_i, t), R(t, t + T_f)\}$ FOR DIFFERENT LEADS T_f AND LAGS FIXED AT $T_i = 1, 6$, WITH $\tau > \hat{\tau}$, $\hat{\alpha}_G \neq \alpha_G$, AND $\hat{\sigma}_G \neq \sigma_G$.

As our examples show, the term structure of the return exhibits different patterns depending on the parameter values. Clearly, from theorem 4, the expected holding-period return $E_t\{R(t, t + T)\}$ always starts from zero when $T = 0$ and converges

to a constant level as the horizon T increases. We also analytically proved and provided the details in Proposition 5 in Appendix A.12 that the derivative of $E_t\{R(t, t + T)\}$ with respect to T does not change its sign more than once. There are therefore four possible patterns: (1) only momentum, (2) only mean-reversion, (3) first mean-reversion and then momentum, (4) first momentum and then mean-reversion. The last pattern is by-and-large consistent with empirical findings of short-run momentum and long-run mean-reversion. It would be interesting for future research to see whether our structural model can generate quantitatively realistic patterns of returns.

6. Conclusion

Defining the concept of market efficiency is not straightforward. In a realistic setting with heterogeneous beliefs and private information, it is not obvious how to define “the market” so that it makes sense to say that the market uses information correctly.

In our dynamic model with traders who solve complicated inference problems, even though the prices fully reflect the average signal at each point in time, traders regularly spot profit opportunities and think they can make money at the expense of others. Except for a very special set of beliefs, the economist also finds profit opportunities. In our paper, the market is efficient if efficiency is defined as fully revealing prices and it is inefficient if efficiency is defined as absence of profit opportunities.

A minimal structural model of anomalies should be a dynamic steady state model with parameters governing the risk-free rate, volatility, mean reversion of both cash flows and dividend growth, asset supply, and the risk aversion of investors. In order to model private information, parameters are needed for the number of informed traders and the precision of their signals. Our model thus has this minimal set of necessary parameters.

The model makes it possible to test the non-behavioral predictions of the weak rational expectations hypothesis about expected returns. Different choices of model parameters generate patterns of momentum and mean reversion over different horizons. Calibrating the model to study whether it can generate quantitatively realistic return dynamics and quantitatively realistic trading behavior presents interesting issues for future research.

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A. Proofs

A.1. Proof of Theorem 1

To solve the equilibrium, we conjecture that price is a linear function of $D(t)$ and $\bar{G}(t)$, specifically,

$$(A-1) \quad P(t) = \frac{D(t)}{r + \alpha_D} + C_G \frac{\bar{G}(t)}{(r + \alpha_D)(r + \alpha_G)}.$$

Define

(A-2)

$$H_n^c(t) := H_n(t) + \hat{A}H_0(t), \quad H_{-n}^c(t) := H_{-n}(t) + \hat{A}H_0(t), \quad \hat{A} := \frac{\tau_0^{1/2}}{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}.$$

It can be shown that

(A-3)

$$\begin{aligned} dP(t) = & -\frac{1}{r + \alpha_D} \left(\alpha_D D(t) - \sigma_G \Omega^{1/2} \left(\tau_H^{1/2} H_n^c(t) + (N-1)\tau_L^{1/2} H_{-n}^c(t) \right) \right) dt \\ & + \frac{C_G \sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r + \alpha_D)(r + \alpha_G)} \left((a_1 + (N-1)a_4) H_n^c(t) + (a_3 + (N-1)a_2) H_{-n}^c(t) \right) dt \\ & + \frac{1}{r + \alpha_D} \left((G^*(t) - G_n(t)) dt + \sigma_D dB_D \right) \\ & + \frac{C_G \sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r + \alpha_D)(r + \alpha_G)} \left(N \hat{A} dB_0^n(t) + dB_n^n(t) + \sum_{m=1; m \neq n} dB_m^n(t) \right). \end{aligned}$$

We conjecture and verify that the value function $V(W_n, H_n^c, H_{-n}^c)$ has the specific quadratic exponential form

(A-4)

$$V(W_n, H_n^c, H_{-n}^c) = -\exp \left(\psi_0 + \psi_W W_n + \frac{1}{2} \psi_{nn} (H_n^c)^2 + \frac{1}{2} \psi_{xx} (H_{-n}^c)^2 + \psi_{nx} H_n^c H_{-n}^c \right).$$

The five constants ψ_0 , ψ_W , ψ_{nn} , ψ_{xx} , and ψ_{nx} have values consistent with a steady-state equilibrium. The terms ψ_{nn} , ψ_{xx} , and ψ_{nx} capture the value of future trading opportunities based on current public and private information. The value of trading on innovations to future information is built into the constant term ψ_0 .

Define constants a_1 , a_2 , a_3 , and a_4 by

$$(A-5) \quad \begin{aligned} a_1 &:= -\alpha_G - \tau + \tau_H^{1/2} (\tau_H^{1/2} + \hat{A}\tau_0^{1/2}), \\ a_2 &:= -\alpha_G - \tau + (N-1)\tau_L^{1/2} (\tau_L^{1/2} + \hat{A}\tau_0^{1/2}), \\ a_3 &:= (\tau_H^{1/2} + \hat{A}\tau_0^{1/2})(N-1)\tau_L^{1/2}, \\ a_4 &:= (\tau_L^{1/2} + \hat{A}\tau_0^{1/2})\tau_H^{1/2}. \end{aligned}$$

The Hamilton-Jacobi-Bellman (HJB) equation corresponding to the conjectured value function $V(W_n, H_n^c, H_{-n}^c)$ in equation (A-4) is

(A-6)

$$\begin{aligned}
0 = \min_{c_n, s_n} & -\frac{e^{-Ac_n}}{V} - \rho + \psi_W \left(rW_n + S_n D(t) - c_n - rP(t)S_n(t) - \frac{\alpha_D}{r + \alpha_D} D(t)S_n \right. \\
& + \frac{\sigma_G \Omega^{1/2}}{r + \alpha_D} \left(\tau_H^{1/2} H_n^c(t) + (N-1)\tau_L^{1/2} H_{-n}^c(t) \right) S_n \\
& + \frac{C_G \sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2})}{N(r + \alpha_D)(r + \alpha_G)} \left((a_1 + (N-1)a_4)H_n^c(t) + (a_3 + (N-1)a_2)H_{-n}^c(t) \right) S_n \left. \right) \\
& + (\psi_{nn}H_n^c(t) + \psi_{nx}H_{-n}^c(t)) \left(-(\alpha_G + \tau)H_n^c + (\tau_H^{1/2} + \hat{A}\tau_0^{1/2})(\tau_H^{1/2}H_n^c + (N-1)\tau_L^{1/2}H_{-n}^c) \right) \\
& + (\psi_{xx}H_{-n}^c(t) + \psi_{nx}H_n^c(t)) \left(-(\alpha_G + \tau)H_{-n}^c + (\tau_L^{1/2} + \hat{A}\tau_0^{1/2})(\tau_H^{1/2}H_n^c + (N-1)\tau_L^{1/2}H_{-n}^c) \right) \\
& + \frac{1}{2}\psi_W^2 S_n^2 \left(\frac{C_G^2 \sigma_G^2 \Omega (N\hat{A}^2 + 1)(\tau_H^{1/2} + (N-1)\tau_L^{1/2})^2}{N(r + \alpha_D)^2(r + \alpha_G)^2} + \frac{\sigma_D^2}{(r + \alpha_D)^2} + \frac{2C_G \sigma_G \sigma_D \Omega^{1/2} \tau_0^{1/2}}{(r + \alpha_D)^2(r + \alpha_G)} \right) \\
& + \frac{1}{2} \left((\psi_{nn}H_n^c(t) + \psi_{nx}H_{-n}^c(t))^2 + \psi_{nn} \right) \left(1 + \hat{A}^2 \right) \\
& + \frac{1}{2} \left((\psi_{xx}H_{-n}^c(t) + \psi_{nx}H_n^c(t))^2 + \psi_{xx} \right) \left(\frac{1}{N-1} + \hat{A}^2 \right) \\
& + \psi_W S_n \left((\psi_{nn} + \psi_{nx})H_n^c(t) + (\psi_{xx} + \psi_{nx})H_{-n}^c(t) \right) \\
& \quad \cdot \left(\frac{C_G \sigma_G \Omega^{1/2}}{N(r + \alpha_D)(r + \alpha_G)} (\tau_H^{1/2} + (N-1)\tau_L^{1/2})(N\hat{A}^2 + 1) + \frac{\sigma_D \hat{A}}{r + \alpha_D} \right) \\
& + \left((\psi_{nn}H_n^c(t) + \psi_{nx}H_{-n}^c(t)) (\psi_{xx}H_{-n}^c(t) + \psi_{nx}H_n^c(t)) + \psi_{nx} \right) \hat{A}^2.
\end{aligned}$$

The solution for optimal consumption is

$$(A-7) \quad c_n^*(t) = -\frac{1}{A} \log \left(\frac{\psi_W V(t)}{A} \right).$$

Plugging optimal consumption and $P(t)$ from equation (A-1) into the HJB equation yields a quadratic function of S_n . It can be shown that the optimal trading strategy

is a linear function of the state variables $H_n^c(t)$ and $H_{-n}^c(t)$,

$$(A-8) \quad S_n^*(t) = C \left(C_G \sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2}) \right. \\ \cdot \left((r - a_1 - (N-1)a_4) H_n^c(t) + ((N-1)(r - a_2) - a_3) H_{-n}^c(t) \right) \\ - \sigma_G \Omega^{1/2} (r + \alpha_G) N \left(\tau_H^{1/2} H_n^c(t) + (N-1)\tau_L^{1/2} H_{-n}^c(t) \right) \\ - \left((\psi_{nn} + \psi_{nx}) H_n^c(t) + (\psi_{xx} + \psi_{nx}) H_{-n}^c(t) \right) \\ \cdot \left. \left(C_G \sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2}) (N\hat{A}^2 + 1) + \sigma_D \hat{A} N (r + \alpha_G) \right) \right),$$

where

$$(A-9) \quad C = \frac{(r + \alpha_D)(r + \alpha_G)/\psi_W}{C_G^2 \sigma_G^2 \Omega (\tau_H^{1/2} + (N-1)\tau_L^{1/2})^2 (N\hat{A}^2 + 1) + N\sigma_D^2 (r + \alpha_G)^2 + 2N(r + \alpha_G)\sigma_D C_G \sigma_G \Omega^{1/2} \tau_0^{1/2}}.$$

Since the market clears, $\sum_{n=1}^N S_n^*(t) = 0$, this implies

$$(A-10) \quad C_G = \frac{N(r + \alpha_G) \left(\sigma_G \Omega^{1/2} + \sigma_D \hat{A} (\psi_{nn} + \psi_{xx} + 2\psi_{nx}) / (\tau_H^{1/2} + (N-1)\tau_L^{1/2}) \right)}{\sigma_G \Omega^{1/2} \left(N(r + \alpha_G) + (N-1)(\tau_H^{1/2} - \tau_L^{1/2})^2 - (1 + N\hat{A}^2)(\psi_{nn} + \psi_{xx} + 2\psi_{nx}) \right)}.$$

Combining equations (A-8) and (A-10), we get

$$(A-11) \quad S_n^*(t) = C_L (H_n^c(t) - H_{-n}^c(t)),$$

where the constant C_L is defined as

$$(A-12) \quad C_L = C \left(\sigma_G \Omega^{1/2} \left(C_G (\tau_H^{1/2} + (N-1)\tau_L^{1/2}) (r - a_1 - (N-1)a_4) - N\tau_H^{1/2} (r + \alpha_G) \right) \right. \\ \left. - (\psi_{nn} + \psi_{nx}) \left(C_G \sigma_G \Omega^{1/2} (\tau_H^{1/2} + (N-1)\tau_L^{1/2}) (1 + N\hat{A}^2) + \sigma_D \hat{A} N (r + \alpha_G) \right) \right).$$

Plugging (A-7) and (A-11) back into the Bellman equation and setting the constant term and the coefficients of W_n , $(H_n^c)^2$, $(H_{-n}^c)^2$, and $H_n^c H_{-n}^c$ to be zero, we obtain five equations, from which we can find five unknown parameters $\psi_0, \psi_W, \psi_{nn}, \psi_{nx}$ and ψ_{xx} .

By setting the constant term and coefficient of W_n to be zero, we obtain

$$(A-13) \quad \psi_W = -rA,$$

$$(A-14) \quad \psi_0 = 1 - \log(r) + \frac{1}{r} \left(-\rho + \frac{1}{2}(1 + \hat{A}^2)\psi_{nn} + \frac{1}{2} \left(\frac{1}{N-1} + \hat{A}^2 \right) \psi_{xx} + \hat{A}^2 \psi_{nx} \right).$$

By setting the coefficients of $(H_n^c)^2$, $(H_{-n}^c)^2$ and $H_n^c H_{-n}^c$ to be zero, we obtain three polynomial equations in the three unknowns ψ_{nn} , ψ_{xx} , and ψ_{nx} . Defining c_1 , c_2 , c_3 , and c_4 by

$$(A-15) \quad c_1 = \frac{C_G^2 \sigma_G^2 \Omega (N \hat{A}^2 + 1) (\tau_H^{1/2} + (N-1) \tau_L^{1/2})^2}{N(r + \alpha_D)^2 (r + \alpha_G)^2} + \frac{\sigma_D^2}{(r + \alpha_D)^2} + \frac{2C_G \sigma_G \sigma_D \Omega^{1/2} \tau_0^{1/2}}{(r + \alpha_D)^2 (r + \alpha_G)},$$

$$(A-16) \quad c_2 = \frac{C_G \sigma_G \Omega^{1/2}}{N(r + \alpha_D)(r + \alpha_G)} (\tau_H^{1/2} + (N-1) \tau_L^{1/2}) (N \hat{A}^2 + 1) + \frac{\sigma_D \hat{A}}{r + \alpha_D},$$

$$(A-17) \quad c_3 = \frac{r A \sigma_G \Omega^{1/2} C_L}{r + \alpha_D} \left(\frac{C_G (\tau_H^{1/2} + (N-1) \tau_L^{1/2}) (r - a_1 - (N-1) a_4)}{N(r + \alpha_G)} - \tau_H^{1/2} \right),$$

$$(A-18) \quad c_4 = \frac{r A \sigma_G \Omega^{1/2} C_L}{r + \alpha_D} \left(\frac{C_G (\tau_H^{1/2} + (N-1) \tau_L^{1/2}) (r - a_2 - \frac{a_3}{N-1})}{N(r + \alpha_G)} - \tau_L^{1/2} \right),$$

these three equations in three unknowns can be written as follows:

$$(A-19) \quad \begin{aligned} & \underline{(H_n^c)^2} : \\ 0 = & -\frac{r}{2} \psi_{nn} + a_1 \psi_{nn} + a_4 \psi_{nx} - r A C_L c_2 (\psi_{nn} + \psi_{nx}) + \frac{1}{2} (1 + \hat{A}^2) \psi_{nn}^2 \\ & + \frac{1}{2} \left(\frac{1}{N-1} + \hat{A}^2 \right) \psi_{nx}^2 + \hat{A}^2 \psi_{nn} \psi_{nx} + c_3 + \frac{1}{2} r^2 A^2 c_1 C_L^2, \end{aligned}$$

$$(A-20) \quad \begin{aligned} & \underline{(H_{-n}^c)^2} : \\ 0 = & -\frac{r}{2} \psi_{xx} + a_2 \psi_{xx} + a_3 \psi_{nx} + r A C_L c_2 (\psi_{xx} + \psi_{nx}) + \frac{1}{2} (1 + \hat{A}^2) \psi_{xx}^2 \\ & + \frac{1}{2} \left(\frac{1}{N-1} + \hat{A}^2 \right) \psi_{xx}^2 + \hat{A}^2 \psi_{xx} \psi_{nx} - (N-1) c_4 + \frac{1}{2} r^2 A^2 c_1 C_L^2, \end{aligned}$$

$$(A-21) \quad \begin{aligned} & \underline{H_n^c H_{-n}^c} : \\ 0 = & -r \psi_{nx} + (a_1 + a_2) \psi_{nx} + a_3 \psi_{nn} + a_4 \psi_{xx} + r A C_L c_2 (\psi_{nn} - \psi_{xx}) \\ & + (1 + \hat{A}^2) \psi_{nn} \psi_{nx} + \left(\frac{1}{N-1} + \hat{A}^2 \right) \psi_{xx} \psi_{nx} + \hat{A}^2 (\psi_{nn} \psi_{xx} + \psi_{nx}^2) \\ & + (N-1) c_4 - c_3 - r^2 A^2 c_1 C_L^2. \end{aligned}$$

To summarize, the optimal consumption is defined in (A-7), the optimal strategy is defined in (A-11) and the endogenous coefficient C_L is defined in (A-12). The equilibrium price is defined in (A-1) and the endogenous coefficient C_G is defined in (A-10). Parameters ψ_W and ψ_0 are presented in (A-13) and (A-14). Parameters ψ_{nn} , ψ_{nx} , ψ_{xx} are solved numerically from the system of the three equations (A-19)-(A-21). These results are presented in Theorem 1.

A.2. Proof of Proposition 1

Assume $\tau_H > \tau_L$. Since information cannot have negative value in the value function (A-4) (since traders can ignore it), the 2×2 matrix

$$(A-22) \quad \begin{pmatrix} \psi_{nn} & \psi_{nx} \\ \psi_{nx} & \psi_{xx} \end{pmatrix}$$

must be negative semi-definite. This implies $\psi_{nn} \leq 0$, $\psi_{xx} \leq 0$, and $\psi_{nx}^2 \leq \psi_{nn}\psi_{xx}$. It follows that $\psi_{nn} + \psi_{xx} + 2\psi_{nx} \leq 0$. Then from equation (A-10), we have that

$$(A-23) \quad C_G \leq (1 + (1 - 1/N)(\tau_H^{1/2} - \tau_L^{1/2})^2 / (r + \alpha_G))^{-1} < 1.$$

Jensen's inequality implies that $0 < C_J < 1$.

Suppose $\tau_H = \tau_L$, then clearly $\psi_{nn} = \psi_{nx} = \psi_{xx} = 0$ solves the three equations (A-19)–(A-21), and additionally we get $C_G = 1$ and $C_L = 0$ from equations (A-10) and (A-12). There is no trading.

A.3. Dampening Effect: The Present Value of Expected Cumulative Dividends and Cash Flow

In this section, we discuss expectations of each trader about how his own valuation, the average valuation of other traders, and how the market price will evolve over time. The discussion here follows Kyle, Obizhaeva and Wang (2016), with some changes to accommodate differences between a setting with imperfect competition and the competitive setting of this paper. In line with Samuelson (1965), the trader's own valuation is a martingale with respect to the trader's own filtration. Each trader believes the average valuation of other traders and market prices follow a more complicated dynamics.

Define the $N + 1$ processes dB_0^n , dB_n^n , and dB_m^n , $m = 1, \dots, N$, $m \neq n$, by

$$(A-24) \quad dB_0^n(t) = \tau_0^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_D(t),$$

$$(A-25) \quad dB_n^n(t) = \tau_H^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_n(t),$$

and

$$(A-26) \quad dB_m^n(t) = \tau_L^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_m(t).$$

The superscript n indicates conditioning on beliefs of trader n . Since trader n 's forecast of the error $G^*(t) - G_n(t)$ is zero given his information set, these $N + 1$ processes are independently-distributed Brownian motions from the perspective of

trader n . In terms of these Brownian motions, trader n believes that signals change as follows:

$$(A-27) \quad dH_0(t) = -(\alpha_G + \tau) H_0(t) dt + \tau_0^{1/2} \frac{G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_0^n(t),$$

$$(A-28) \quad dH_n(t) = -(\alpha_G + \tau) H_n(t) dt + \tau_H^{1/2} \frac{G_n(t)}{\sigma_G \Omega^{1/2}} dt + dB_n^n(t),$$

$$(A-29) \quad dH_{-n}(t) = -(\alpha_G + \tau) H_{-n}(t) dt + \tau_L^{1/2} \frac{G_n(t)}{\sigma_G \Omega^{1/2}} dt + \frac{1}{N-1} \sum_{\substack{m=1 \\ m \neq n}}^N dB_m^n(t).$$

Note that the coefficient $\tau_H^{1/2}$ in the second term on the right hand side of equation (A-28) is different from the coefficient $\tau_L^{1/2}$ in the second term on the right hand side of equation (A-29). This difference is the key driving force behind the price-dampening effect resulting from the Keynesian beauty contest. It captures the fact that—in addition to disagreeing about the value of the asset in the present—traders also disagree about the dynamics of their future valuations.

We can derive the stochastic process for $G_n(t)$ and $G_{-n}(t) := \frac{1}{N-1} \sum_{m=1, \dots, N; m \neq n} G_m(t)$ as follows:

$$(A-30) \quad dG_n(t) = -\alpha_G G_n(t) dt + \sigma_G \Omega^{1/2} \left(\tau_0^{1/2} dB_0^n(t) + \tau_H^{1/2} dB_n^n(t) + \tau_L^{1/2} \sum_{\substack{m=1 \\ m \neq n}}^N dB_m^n(t) \right),$$

$$(A-31) \quad dG_{-n}(t) = -(\alpha_G + \tau) G_{-n}(t) dt + \left(\tau_0 + \tau_L^{1/2} \left(2\tau_H^{1/2} + (N-2)\tau_L^{1/2} \right) \right) G_n(t) dt \\ + \sigma_G \Omega^{1/2} \left(\tau_0^{1/2} dB_0^n(t) + \tau_L^{1/2} dB_n^n(t) + \frac{\tau_H^{1/2} + (N-2)\tau_L^{1/2}}{N-1} \sum_{\substack{m=1 \\ m \neq n}}^N dB_m^n(t) \right).$$

From (A-31), when $G_m(t) = G_n(t)$, trader n believes that other traders' estimates of expected growth rates $G_m(t)$ will mean-revert to zero at a rate $\alpha_G + (\tau_H^{1/2} - \tau_L^{1/2})^2 > \alpha_G$. From (A-30), trader n believes that his own estimate of expected growth rate $G_n(t)$ will mean-revert to zero at a rate α_G .

From (A-30), (A-31), and (12), the expected dynamics of $G_n(t)$, $G_{-n}(t)$, and $D(t)$ are given by

$$(A-32) \quad \mathbb{E}_0^n \{G_n(t)\} = e^{-\alpha_G t} G_n(0),$$

$$(A-33) \quad \mathbb{E}_0^n \{G_{-n}(t)\} = \frac{1}{\tau} \left(\tau_0 + \tau_L^{1/2} \left(2\tau_H^{1/2} + (N-2)\tau_L^{1/2} \right) \right) (e^{-\alpha_G t} - e^{-(\alpha_G + \tau)t}) G_n(0) + e^{-(\alpha_G + \tau)t} G_{-n}(0),$$

$$(A-34) \quad \mathbb{E}_0^n \{D(t)\} = \frac{1}{\alpha_D - \alpha_G} (e^{-\alpha_G t} - e^{-\alpha_D t}) G_n(0) + e^{-\alpha_D t} D(0).$$

The present value of expected cumulative dividends and cash flow from liquidating one share of the stock at date t using trader n 's estimate of fundamental value is

$$(A-35) \quad PV_n(0, t) := \mathbb{E}_0^n \left\{ \int_0^t e^{-ru} D(u) du + e^{-rt} \left(\frac{D(t)}{r + \alpha_D} + \frac{G_n(t)}{(r + \alpha_D)(r + \alpha_G)} \right) \right\}.$$

Substituting (A-32) and (A-34) into (A-35), it can be shown that (A-35) is equal to

$$(A-36) \quad PV_n(0, t) = F_n(0) = \frac{D(0)}{r + \alpha_D} + \frac{G_n(0)}{(r + \alpha_D)(r + \alpha_G)}.$$

The present value of expected cumulative dividends and cash flow from liquidating one share of the stock at date t using others' valuations $\sum_{m \neq n} F_m(t)/(N-1)$ is

$$(A-37) \quad PV_{-n}(0, t) := \mathbb{E}_0^n \left\{ \int_0^t e^{-ru} D(u) du + e^{-rt} \left(\frac{D(t)}{r + \alpha_D} + \frac{G_{-n}(t)}{(r + \alpha_D)(r + \alpha_G)} \right) \right\}.$$

Assuming $G_m(0) = G_n(0) = \bar{G}(0)$ and substituting (A-32)–(A-34) into (A-37), it can be shown that equation (A-37) is equal to

$$(A-38) \quad PV_{-n}(0, t) = F_n(0) + \frac{(\tau_H^{1/2} - \tau_L^{1/2})^2}{\tau(r + \alpha_G)(r + \alpha_D)} (e^{-(r + \alpha_G + \tau)t} - e^{-(r + \alpha_G)t}) G_n(0).$$

Similarly, the present value of expected cumulative dividends and cash flow from liquidating one share of the stock at date t at the equilibrium price $P(t)$ is

$$(A-39) \quad PV_p(0, t) := \mathbb{E}_0^n \left(\int_0^t e^{-ru} D(u) du + e^{-rt} \left(\frac{D(t)}{r + \alpha_D} + \frac{C_G \bar{G}(t)}{(r + \alpha_D)(r + \alpha_G)} \right) \right).$$

Substituting (A-32)–(A-34) into (A-39), it can be shown that (A-39) is equivalent to

$$(A-40) \quad PV_p(0, t) = F_n(0) + \frac{C_G \left(N - (\tau_H^{1/2} - \tau_L^{1/2})^2 \tau^{-1} (N-1) \right) - N}{N(r + \alpha_G)(r + \alpha_D)} e^{-(r + \alpha_G)t} G_n(0) \\ + \frac{C_G (\tau_H^{1/2} - \tau_L^{1/2})^2 \tau^{-1} (N-1)}{N(r + \alpha_G)(r + \alpha_D)} e^{-(r + \alpha_G + \tau)t} G_n(0).$$

We now use Figure 10 to illustrate the intuition behind the dampening effect.⁷ Figure 10 depicts a graph of the three functions $PV_n(0, t)$, $PV_{-n}(0, t)$, and $PV_p(0, t)$ with time t on the horizontal axis and the results of three different present value calculations on the vertical axis. For simplicity of exposition, assume that the buy-and-hold valuations of all N traders coincide at time 0, and these estimates are positive; specifically, assume that for all n , we have $G_n(0) = G_{-n}(0) = \bar{G}(0) > 0$. For negative values, the figure will be symmetric. Details for the present value calculations are given in equations (A-36), (A-38), and (A-40). By assumption, these three calculations are done using trader n 's beliefs, but they are identical for all traders.

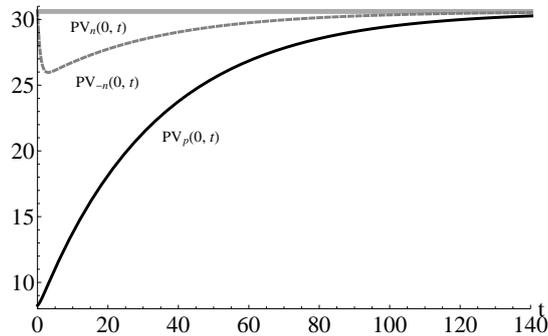


FIGURE 10. PRESENT VALUE OF DIVIDENDS AND LIQUIDATION VALUE FROM THE PERSPECTIVE OF A TRADER.

The horizontal light solid line is based on the assumption that trader n liquidates the asset at date t at a valuation equal to his own estimate of its fundamental value $F_n(t)$. Since trader n applies Bayes law correctly given his beliefs, the martingale property of his valuation (law of iterated expectations) makes the present value $PV_n(0, t)$ a constant function for $t \geq 0$; its graph is a horizontal line.

The light dashed curve is based on the assumption that trader n liquidates the asset at a valuation equal to the average estimate of fundamental value of the other $N - 1$ traders. The N traders' estimates of fundamental value are the same at date 0. Due to disagreement about signal precision, trader n believes that the other $N - 1$ traders' estimates of the growth rate $G^*(t)$ will mean revert to zero at rate $\alpha_G + (\tau_H^{1/2} - \tau_L^{1/2})^2$, which is faster than the mean reversion rate α_G he assumes for his own forecast.

We next provide results which calculate the derivative of the present value of

⁷Numerical calculations are based on the parameter values $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $N = 10$, $G_n(0) = 0.08$, $D(0) = 0.7$, $\tau_H = 0.8$, $\tau_L = 0.09$, and $\tau_0 = 0.024$.

cash flows $PV_{-n}(0, t)$ with respect to time. From (A-37), it follows that

$$(A-41) \quad \frac{dPV_{-n}(0, t)}{dt} = \frac{(\tau_H^{1/2} - \tau_L^{1/2})^2 G_n(0) e^{-(r+\alpha_G)t}}{\tau(r + \alpha_G)(r + \alpha_D)} \left((r + \alpha_G) - (r + \alpha_G + \tau)e^{-\tau t} \right).$$

(A-41) implies that $dPV_{-n}(0, t)/dt < 0$ iff $t < \frac{1}{\tau} \ln \left(1 + \frac{\tau}{r + \alpha_G} \right)$.

Intuitively, as a result of the higher mean-reversion rate, trader n believes that $PV_{-n}(0, t)$ will fall in the short run. Since trader n believes that his own initial present value calculation is correct, trader n believes that $PV_{-n}(0, t)$ will rise back to his own estimate of the fundamental value in the long run. Thus, the graph depicted by the dashed line in figure 10 first falls below the horizontal line in the short run and then rises asymptotically back toward it in the long run.

The dark solid curve is based on the assumption that trader n liquidates the asset at a valuation equal to his estimate of the equilibrium market price $P(t)$. Let $PV_p(0, t)$ denote the result of this present value calculation. Consistent with the equilibrium result $0 < C_G < 1$, the initial price $P(0)$ is lower than the consensus fundamental value, even though all traders by assumption agree about this current fundamental value, agree about how it will evolve in the future, and know that they agree with the valuation dynamics. The dampening effect nevertheless arises due to interactions among expectations of traders in our model. If prices were equal to the consensus fundamental valuation, all traders would want to hold short positions because all of them would expect prices to fall below fundamental value in the short run as the others temporarily became more bearish. As a result, the price $P(0)$ is dampened relative to the average fundamental valuation in the market; yet this is consistent with each trader having a target inventory of zero at date 0.

We now calculate the derivative of the present value of cash flows $PV_p(0, t)$ with respect to time. From (A-40), it follows that

$$(A-42) \quad \frac{dPV_p(0, t)}{dt} = \frac{G_n(0) e^{-(r+\alpha_G)t}}{N(r + \alpha_G)(r + \alpha_D)} \left(\left(N - C_G \left(N - (\tau_H^{1/2} - \tau_L^{1/2})^2 \tau^{-1} (N - 1) \right) \right) (r + \alpha_G) \right. \\ \left. C_G (\tau_H^{1/2} - \tau_L^{1/2})^2 \tau^{-1} (N - 1) (r + \alpha_G + \tau) e^{-\tau t} \right).$$

Clearly, (A-42) implies $dPV_p(0, t)/dt \rightarrow 0$ when $t \rightarrow \infty$. Define

$$(A-43) \quad \hat{t} := -\frac{1}{\tau} \ln \left(\left(1 + \frac{(1 - C_G)N\tau}{C_G(\tau_H^{1/2} - \tau_L^{1/2})^2(N - 1)} \right) \frac{r + \alpha_G}{r + \alpha_G + \tau} \right).$$

Equation (A-42) implies $dPV_p(0, t)/dt > 0$ if and only if $t > \hat{t}$. It can be shown that $\hat{t} > 0$ if and only if $C_G > \hat{C}_G := \left(1 + (1 - 1/N)(\tau_H^{1/2} - \tau_L^{1/2})^2 / (r + \alpha_G) \right)^{-1}$. This further yields the following results:

- If $C_G \leq \hat{C}_G$, then $dPV_p(0, t)/dt > 0$ for all $t > 0$.
- If $C_G > \hat{C}_G$, then $dPV_p(0, t)/dt = 0$ for $t = \hat{t}$, $dPV_p(0, t)/dt > 0$ for $t > \hat{t}$, and $dPV_p(0, t)/dt < 0$ for $t < \hat{t}$.

From Proposition 1, $C_G \leq \hat{C}_G$, therefore, $PV_p(0, t)$ increases monotonically over time. As figure 10 illustrates, trader n expects prices to increase monotonically from a dampened value toward his estimate of fundamental value.

A.4. One-Period Model

A risky asset with random liquidation value $v \sim N(0, 1/\tau_v)$ is traded for a safe numeraire asset. Each of N traders $n = 1, \dots, N$ is endowed with S_n shares of a zero-net-supply risky asset, implying $\sum_{n=1}^N S_n = 0$. Traders observe signals about the normalized liquidation value $\tau_v^{1/2}v$. All traders observe a public signal $i_0 := \xi_0(\tau_0^{1/2}(\tau_v^{1/2}v) + e_0)$ with $e_0 \sim N(0, 1)$. Each trader n observes a private signal $i_n := \xi_n(\tau_n^{1/2}(\tau_v^{1/2}v) + e_n)$ with $e_n \sim N(0, 1)$. The asset payoff v , the public signal error e_0 , and N private signal errors e_1, \dots, e_N are independently distributed.

Traders agree about the precision of the public signal τ_0 and agree to disagree about the precisions of private signals τ_n . Each trader is “relatively overconfident,” believing his own signal has a high precision $\tau_n = \tau_H$ and other traders’ signals have low precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$.

We consider two different cases of information structure. In the first case, $\xi_0 = \tau_0^{-1/2}$, $\xi_n = \tau_H^{-1/2}$, $\xi_m = \tau_L^{-1/2}$; this case corresponds to a conventional modelling approach in the existing microstructure models such as, for example, Kyle (1985) or Allen, Morris and Shin (2006), where information is usually modelled as $v + \epsilon$ with $v \sim N(0, \tau_v^{-1})$ and $\epsilon \sim N(0, \tau_\epsilon^{-1})$; if traders disagree about the precision of information, then they usually disagree about the parameter τ_ϵ .

In the second case, $\xi_0 = \xi_n = \xi_m = 1$; this case more closely corresponds to a dynamic model in our paper where information is essentially modelled as $\tau_n^{1/2}v + \epsilon$ with $v \sim N(0, \tau_v^{-1})$ and $\epsilon \sim N(0, 1)$; if traders disagree about the precision of information, then they disagree about the parameter τ_n .

In the first case, more precise information is modelled by assigning lower weights to the noise component, whereas in the second case more precise information is modelled by assigning bigger weights to signals. The later is more consistent with a dynamic interpretation, since the disagreement about diffusion variance can be quickly resolved in a dynamic setting.

Each trader submits a demand schedule $X_n(p) := X_n(i_0, i_n, S_n, p)$ to a single-price auction. An auctioneer calculates the market-clearing price $p := p[X_1, \dots, X_N]$.

Trader n ’s terminal wealth is

$$(A-44) \quad W_n := v(S_n + X_n) - pX_n.$$

Each trader maximizes the same expected exponential utility function of wealth $E^n\{-e^{-AW_n}\}$ using his own beliefs about τ_H and τ_L to calculate the expectation.

Trader n maximizes his expected utility, or equivalently he maximizes $E^n\{W_n\} - \frac{1}{2}A \text{Var}^n\{W_n\}$. He chooses the quantity to trade x_n that solves the maximization problem

$$(A-45) \quad \max_{x_n} \left(\frac{\tau_v^{1/2}}{\tau} \left(\frac{\tau_0^{1/2}}{\xi_0} i_0 + \frac{\tau_H^{1/2}}{\xi_n} i_n + \frac{(N-1)\tau_L^{1/2}}{\xi_m} i_{-n} \right) (S_n + x_n) - p x_n - \frac{A}{2\tau} (S_n + x_n)^2 \right),$$

where $\tau = (\text{Var}^n\{v\})^{-1} = \tau_v (1 + \tau_0 + \tau_H + (N-1)\tau_L)$. Then, the first-order condition with respect to x_n yields

$$(A-46) \quad x_n^* = \frac{1}{A} \left(\tau_v^{1/2} \left(\frac{\tau_0^{1/2}}{\xi_0} i_0 + \frac{\tau_H^{1/2}}{\xi_n} i_n + \frac{(N-1)\tau_L^{1/2}}{\xi_m} i_{-n} \right) - p \tau \right) - S_n.$$

The market-clearing condition $\sum_{n=1}^N x_n = 0$ yields the equilibrium price

$$(A-47) \quad p^* = \frac{1}{N} \sum_{n=1}^N E^n\{v\} = \frac{\tau_v^{1/2}}{\tau} \left(\frac{\tau_0^{1/2}}{\xi_0} i_0 + \frac{\tau_H^{1/2}/\xi_n + (N-1)\tau_L^{1/2}/\xi_m}{N} \sum_{n=1}^N i_n \right).$$

Substituting (A-47) into (A-46) yields the equilibrium trading strategy

$$(A-48) \quad x_n^* = \frac{1}{A} \left(1 - \frac{1}{N} \right) \tau_v^{1/2} \left(\frac{\tau_H^{1/2}}{\xi_n} - \frac{\tau_L^{1/2}}{\xi_m} \right) (i_n - i_{-n}) - S_n.$$

The two modelling approaches generate equilibrium prices with strikingly different properties. In the first case with $\xi_0 = \tau_0^{-1/2}$, $\xi_n = \tau_H^{-1/2}$, $\xi_m = \tau_L^{-1/2}$, we have

$$(A-49) \quad p^* = \frac{1}{N} \sum_{n=1}^N E^n\{v\} = \frac{\tau_v^{1/2}}{\tau} \left(\tau_0 i_0 + \frac{\tau_H + (N-1)\tau_L}{N} \sum_{n=1}^N i_n \right).$$

The equilibrium price averages the estimates of traders in an intuitive way. It is equal to the expectation of a fundamental value as if all public and private information is included into information set. Information i_0 is assigned precision τ_0 and each private information i_n is assigned the average precision $\frac{1}{N}(\tau_H + (N-1)\tau_L)$.

In the second case of $\xi_0 = \xi_n = \xi_m = 1$, we get

$$(A-50) \quad p^* = \frac{1}{N} \sum_{n=1}^N E^n\{v\} = \frac{\tau_v^{1/2}}{\tau} \left(\tau_0^{1/2} i_0 + \frac{\tau_H^{1/2} + (N-1)\tau_L^{1/2}}{N} \sum_{n=1}^N i_n \right).$$

The equilibrium price does not average the estimates of traders in an intuitive way. It can still be thought of as the expectation of a fundamental value in a full-information case. Though while information i_0 is assigned precision τ_0 , private

information i_n is not assigned the average precision of $\frac{1}{N}(\tau_H + (N-1)\tau_L)$, but instead gets the precision of $(\frac{1}{N}(\tau_H^{1/2} + (N-1)\tau_L^{1/2}))^2$. Due to Jensen's inequality this imputed precision is lower than the average precision. The same intuition explains the beliefs aggregation effect generating momentum in our paper.

A.5. Proof of Proposition 2

To find another closed-form solution, we set $\tau_L = 0$, and then evaluate the solution in the limit as $N \rightarrow \infty$ and $\hat{A} \rightarrow 0$. We conjecture and verify that $\psi_{nn} = \bar{\psi}_{nn}$, $\psi_{nx} = \bar{\psi}_{nx}$, and $\psi_{xx} = \bar{\psi}_{xx}$, where $\bar{\psi}_{nn}$, $\bar{\psi}_{nx}$, and $\bar{\psi}_{xx}$ are constants that do not depend on N .

Solving the system of equations (A-19)–(A-21) yields

$$(A-51) \quad \bar{\psi}_{nn} = \frac{1}{2} \left(r + 2(\alpha_G + \tau - \tau_H) - \left((r + 2(\alpha_G + \tau - \tau_H))^2 + \frac{4\Omega\sigma_G^2\tau_H}{\sigma_D^2} \right)^{1/2} \right),$$

$$(A-52) \quad \bar{\psi}_{nx} = \frac{\Omega\sigma_G^2\tau_H/\sigma_D^2}{r + 2(\alpha_G + \tau) - \tau_H - \bar{\psi}_{nn}},$$

$$(A-53) \quad \bar{\psi}_{xx} = \frac{1}{r + 2\alpha_G + 2\tau} \left(\bar{\psi}_{nx}^2 - \frac{\Omega\sigma_G^2\tau_H}{\sigma_D^2} \right).$$

Equations (A-10) and (A-12) imply that

$$(A-54) \quad C_G \rightarrow \frac{r + \alpha_G}{r + \alpha_G + \tau} < 1, \quad C_L = \frac{\Omega^{1/2}\sigma_G\tau_H^{1/2}(r + \alpha_D)}{Ar\sigma_D^2}.$$

A.6. Proof of Proposition 3

Let a vector $(\psi_{nn}^*, \psi_{nx}^*, \psi_{xx}^*)$ be a solution to the system (A-19)–(A-21) for exogenous parameters A , σ_D , σ_G , r , α_G , α_D , τ_0 , τ_L , and τ_H . If risk aversion is rescaled by factor F from A to A/F and other exogenous parameters are kept unchanged, then it is straightforward to show that the vector $(\psi_{nn}^*, \psi_{nx}^*, \psi_{xx}^*)$ is still the solution to the system (A-19)–(A-21). From equations (34), (A-10), and (A-12), it then follows that C_L changes to $C_L F$, λ changes to λ/F , but C_G remains the same.

A.7. Proof of Theorem 2

The outline of the proof is as follows. At any point of time, the representative agent must have beliefs such that the equilibrium price

$$(A-55) \quad P(t) = \frac{D(t)}{r + \alpha_D} + C_G \frac{\sigma_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \left(\tau_0^{1/2} H_0(t) + \tau_I^{1/2} \sum_{n=1, \dots, N} H_n(t) \right)$$

coincides with his estimate of the fundamental value

$$(A-56) \quad \check{F}(t) = \frac{D(t)}{r + \alpha_D} + \frac{\check{\sigma}_G \check{\Omega}^{1/2}}{(r + \alpha_D)(r + \check{\alpha}_G)} \left(\check{\tau}_0^{1/2} \check{H}_0(t) + \check{\tau}_I^{1/2} \sum_{n=1, \dots, N} \check{H}_n(t) \right).$$

The fundamental value $\check{F}(t)$ is a version of the Gordon growth formula given the estimate $\check{G}(t)$ of the growth rate in equation (44).

First, the history of signals $\check{H}_n(t)$ in equation (42) must coincide with the history of signals $H_n(t)$ in equation (23). This implies the restriction

$$(A-57) \quad \check{\alpha}_G + \check{\tau} = \alpha_G + \tau.$$

Second, the coefficients of the two random variables in the two equations (A-55) and (A-56) must match. This leads to the two restrictions

$$(A-58) \quad \frac{C_G \sigma_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \tau_0^{1/2} = \frac{\check{\sigma}_G \check{\Omega}^{1/2}}{(r + \alpha_D)(r + \check{\alpha}_G)} \check{\tau}_0^{1/2},$$

$$(A-59) \quad \frac{C_G \sigma_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \tau_I^{1/2} = \frac{\check{\sigma}_G \check{\Omega}^{1/2}}{(r + \alpha_D)(r + \check{\alpha}_G)} \check{\tau}_I^{1/2}.$$

The definition of $\check{\tau}_0$ and equation (45) yield the last restriction

$$(A-60) \quad \check{\sigma}_G = \sigma_D \check{\tau}_0^{1/2} (2 \check{\alpha}_G + \check{\tau})^{1/2}.$$

The solution of the system is the set of three parameters $\check{\alpha}_G$, $\check{\sigma}_G$, and $\check{\tau}_I$ describing beliefs of the representative agent stated in the theorem as well as the expression for $\check{\tau}_0$

$$(A-61) \quad \check{\tau}_0 = \tau_0 \frac{C_G (r + \alpha_G + \tau)}{r + \alpha_G + C_G (\tau_0 + N \tau_I)}.$$

Since $N \tau_I < \tau_H + (N - 1)\tau_L$, we have $\check{\alpha}_G > \alpha_G$. Since

$$(A-62) \quad C_G < \left(1 + \frac{N - 1}{N} \frac{(\tau_H^{1/2} - \tau_L^{1/2})^2}{r + \alpha_G} \right)^{-1},$$

we have $\check{\tau}_I < \tau_I$.

A.8. Proof of Theorem 3

From direct calculation, the uncertainty term $d\hat{B}_r(t)$ in equation (63) is defined as

$$(A-63) \quad d\hat{B}_r(t) := \frac{\sigma_G C_G \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \left(\tau_0^{1/2} dB_0^*(t) + \tau_I^{1/2} N d\bar{B}^*(t) \right) + \frac{\sigma_D}{(r + \alpha_D)} dB_0^*(t).$$

The processes $d\bar{B}^*(t)$ and $dB_0^*(t)$, defined as

$$(A-64) \quad d\bar{B}^*(t) := \hat{\tau}_I^{1/2} (\hat{\sigma}_G \hat{\Omega}^{1/2})^{-1} (G^*(t) - \hat{G}(t)) dt + \frac{1}{N} \sum_{n=1}^N d\hat{B}_n(t),$$

$$(A-65) \quad dB_0^*(t) := \hat{\tau}_0^{1/2} (\hat{\sigma}_G \hat{\Omega}^{1/2})^{-1} (G^*(t) - \hat{G}(t)) dt + dB_0(t),$$

are Brownian motions under the empirically correct beliefs. Note that the variance of $dB_0^*(t)$ is equal to one, but the variance of $d\bar{B}^*(t)$ is equal to $1/N$ per unit of time.

From equation (28), we obtain

$$(A-66) \quad H(t) = \left(P(t) - \frac{D(t)}{r + \alpha_D} \right) \frac{(r + \alpha_D)(r + \alpha_G)}{C_G \sigma_G \Omega^{1/2}}.$$

We also have the following relationship between traders' signals $H_n(t)$ and the economist's model of the traders' signal and $\hat{H}_n(t)$, $n = 0, 1, \dots, N$:

$$(A-67) \quad \hat{H}_n(t) = H_n(t) + (\alpha_G + \tau - \hat{\alpha}_G - \hat{\tau}) \int_{u=-\infty}^t e^{-(\hat{\alpha}_G + \hat{\tau})(t-u)} H_n(u) du.$$

Substituting (A-66) and (A-67) into (63) yields (69), where α_1, α_2 , and α_3 are defined as

$$(A-68) \quad \alpha_1 := \left(b \frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} - a \right) \frac{(r + \alpha_D)(r + \alpha_G)}{C_G \sigma_G \Omega^{1/2}},$$

$$(A-69) \quad \alpha_2 := b \frac{\hat{\tau}_I^{1/2}}{\tau_I^{1/2}} \frac{(\alpha_G + \tau - \hat{\alpha}_G - \hat{\tau})(r + \alpha_D)(r + \alpha_G)}{C_G \sigma_G \Omega^{1/2}},$$

$$(A-70) \quad \alpha_3 := -b \frac{\tau_I^{1/2} \hat{\tau}_0^{1/2} - \hat{\tau}_I^{1/2} \tau_0^{1/2}}{\tau_I^{1/2}},$$

with a and b defined in equations (64) and (65).

A.9. Proof of Theorem 4

Using the definitions of $H(t)$ and $\hat{H}(t)$ in equations (29) and (59) as well as equations (56), (58), and (60), we can write a continuous 2-vector stochastic process $y(t) = [H(t), \hat{H}(t)]'$ as satisfying the following linear stochastic differential equation:

$$(A-71) \quad dy(t) = K y(t) dt + C_z dZ(t),$$

where K is a 2×2 matrix and C_z is a 2×2 matrix given by

$$(A-72) \quad K = \begin{pmatrix} -\alpha_G - \tau & \tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2} \\ 0 & -\hat{\alpha}_G \end{pmatrix},$$

$$(A-73) \quad C_z = \begin{pmatrix} \tau_0^{1/2} & N \tau_I^{1/2} \\ \hat{\tau}_0^{1/2} & N \hat{\tau}_I^{1/2} \end{pmatrix}.$$

From the perspective of the economist, the vector $dZ(t) = [dB_0^*(t), d\bar{B}^*(t)]'$ is a 2×1 -dimensional Brownian motion, where $dB_0^*(t)$ is a Brownian motion with variance of one defined in equation (A-65) and $d\bar{B}^*(t)$ is a Brownian motion with variance $1/N$ defined in equation (A-64).

Using results about linear continuous-time stochastic processes, we can represent the process $y(t) = [H(t), \hat{H}(t)]'$ in an integral form as

$$(A-74) \quad y(s) = e^{K(s-t)} y(t) + \int_{u=t}^s e^{K(s-u)} C_z dZ(u).$$

It can be also shown that the exponential 2×2 matrix e^{Kt} is given by

$$(A-75) \quad e^{Kt} = \begin{pmatrix} e^{-(\alpha_G + \tau)t} & \frac{\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2}}{\tau + \alpha_G - \hat{\alpha}_G} (e^{-\hat{\alpha}_G t} - e^{-(\alpha_G + \tau)t}) \\ 0 & e^{-\hat{\alpha}_G t} \end{pmatrix}.$$

Plugging e^{Kt} back into equation (A-74), we obtain recursive formulas for the stochastic vector $y(s) = [H(s), \hat{H}(s)]'$ as a function of $y(t) = [H(t), \hat{H}(t)]'$. Using this result, we can express the cumulative holding-period return $R(t, t+T)$ as a linear function of $H(t)$ and $\hat{H}(t)$ as in (72), where

$$(A-76) \quad \bar{B}(t, t+T) := \int_{s=t}^{t+T} \int_{u=s}^{t+T} [-a, b] e^{K(u-s)} C_z du dZ(s) + \int_{s=t}^{t+T} d\hat{B}_r(s).$$

A.10. Variance-Covariance Matrix of $H(t)$ and $\hat{H}(t)$

To plot the term structure of the holding-period return for different $H(t)$ and $\hat{H}(t)$ —normal random variables with unconditional means of zero—we first derive the steady-state unconditional variance-covariance matrix of $H(t)$ and $\hat{H}(t)$. Define the steady-state unconditional variance-covariance matrix of $H(t)$ and $\hat{H}(t)$ as $Q = ((q_{11}, q_{12}), (q_{12}, q_{22}))$. In the steady state, we have

$$(A-77) \quad KQ + QK' + C_z C_z' = 0.$$

It can be shown that

$$(A-78) \quad q_{11} = \frac{\tau_0 + N \tau_I}{2(\alpha_G + \tau)} + \frac{(2\hat{\alpha}_G + \hat{\tau})(\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2})^2}{2\hat{\alpha}_G (\alpha_G + \hat{\alpha}_G + \tau) (\alpha_G + \tau)},$$

$$(A-79) \quad q_{12} = \frac{(2\hat{\alpha}_G + \hat{\tau}) (\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2})}{2\hat{\alpha}_G (\alpha_G + \tau + \hat{\alpha}_G)},$$

$$(A-80) \quad q_{22} = \frac{\hat{\tau}}{2\hat{\alpha}_G}.$$

The distribution of $\hat{H}(t)$ conditional on $H(t)$ is a normal random variable described by the first two moments

$$(A-81) \quad E\{\hat{H}(t) | H(t)\} = q_{12}/q_{11} H(t), \quad \text{Var}\{\hat{H}(t) | H(t)\} = q_{22} - q_{12}^2/q_{11}.$$

To focus on economically relevant ranges, we consider a +1 standard deviation event for $H(t)$, i.e., $H(t) = q_{11}^{1/2}$, and k -standard deviation events for $\hat{H}(t)$ conditional on $H(t)$, i.e., $\hat{H}(t) = q_{12}/q_{11} H(t) + k (q_{22} - q_{12}^2/q_{11})^{1/2}$, where $k = -2, -1, 0, 1, 2$.

A.11. Covariance and Correlation of $R(t - T_l, t)$ and $R(t, t + T_f)$

We now calculate the covariance of $R(t - T_l, t)$ and $R(t, t + T_f)$. Since the unconditional means of $R(t - T_l, t)$ and $R(t, t + T_f)$ are zero, equations (72) and (A-74) yield

$$(A-82) \quad \begin{aligned} \text{Cov}\{R(t - T_l, t), R(t, t + T_f)\} &= E\{R(t - T_l, t) R(t, t + T_f)\} \\ &= E\{(-\beta_1(T_l) H(t - T_l) + \beta_2(T_l) \hat{H}(t - T_l) + \bar{B}(t - T_l, t)) (-\beta_1(T_f) H(t) + \beta_2(T_f) \hat{H}(t))\}, \end{aligned}$$

where $\beta_1(\cdot)$ and $\beta_2(\cdot)$ are as defined in Theorem 4. Define

$$(A-83) \quad \eta := \frac{\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2}}{\tau + \alpha_G - \hat{\alpha}_G}, \quad m_1 := -\frac{a(\hat{\tau}_0^{1/2} \eta - \tau_0^{1/2})^2}{\alpha_G + \tau} \beta_1(T_f),$$

$$(A-84) \quad m_3 := (\hat{\tau}_0^{1/2} \eta - \tau_0^{1/2}) \hat{\tau}_0^{1/2} \left(\frac{-a(\beta_2(T_f) - \eta \beta_1(T_f))}{\alpha_G + \tau} + \frac{-\beta_1(T_f)(-a\eta + b)}{\hat{\alpha}_G} \right),$$

$$(A-85) \quad m_4 := \beta_1(T_f) (\tau_0^{1/2} - \hat{\tau}_0^{1/2} \eta) \left(\frac{\hat{\tau}_0^{1/2} (-a\eta + b)}{\hat{\alpha}_G} + \frac{\sigma_G \Omega^{1/2} C_G \tau_0^{1/2} + \sigma_D (r + \alpha_G)}{(r + \alpha_D) (r + \alpha_G)} + \frac{a(\hat{\tau}_0^{1/2} \eta - \tau_0^{1/2})}{\alpha_G + \tau} \right),$$

$$(A-86) \quad m_2 := -\frac{\hat{\tau}_0 (-a\eta + b)}{\hat{\alpha}_G} (\beta_2(T_f) - \eta \beta_1(T_f)), \quad m_5 := -\frac{(\beta_2(T_f) - \eta \beta_1(T_f)) \hat{\tau}_0^{1/2}}{\beta_1(T_f) (\tau_0^{1/2} - \hat{\tau}_0^{1/2} \eta)} m_4,$$

(A-87)

$$n_1 := -\beta_1(T_f)N \frac{a(\hat{\tau}_I^{1/2}\eta - \tau_I^{1/2})^2}{\alpha_G + \tau}, \quad n_2 := -(-\beta_1(T_f)\eta + \beta_2(T_f)) \frac{N\hat{\tau}_I(-a\eta + b)}{\hat{\alpha}_G},$$

(A-88)

$$n_3 := N\hat{\tau}_I^{1/2}(\hat{\tau}_I^{1/2}\eta - \tau_I^{1/2}) \left(\frac{-a}{\alpha_G + \tau}(-\beta_1(T_f)\eta + \beta_2(T_f)) - \beta_1(T_f) \frac{-a\eta + b}{\hat{\alpha}_G} \right),$$

(A-89)

$$n_4 := -N\beta_1(T_f)(\tau_I^{1/2} - \hat{\tau}_I^{1/2}\eta) \left(\frac{-a(\tau_I^{1/2} - \hat{\tau}_I^{1/2}\eta)}{\alpha_G + \tau} + \frac{\hat{\tau}_I^{1/2}(-a\eta + b)}{\hat{\alpha}_G} + \frac{\sigma_G\Omega^{1/2}C_G\tau_I^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \right),$$

(A-90)

$$n_5 := -\frac{(\beta_2(T_f) - \eta\beta_1(T_f))\hat{\tau}_I^{1/2}}{\beta_1(T_f)(\tau_I^{1/2} - \hat{\tau}_I^{1/2}\eta)} n_4.$$

Then direct calculations show that $\text{Cov}\{R(t - T_l, t), R(t, t + T_f)\}$ is a function of T_l and T_f given by

(A-91)

$$\begin{aligned} & \text{Cov}\{R(t - T_l, t), R(t, t + T_f)\} = \\ & -\beta_1(T_f)(-\beta_1(T_l)(q_{11} - q_{12}\eta) + \beta_2(T_l)(q_{12} - q_{22}\eta))e^{-(\alpha_G + \tau)T_l} \\ & + (\beta_2(T_l)q_{22} - \beta_1(T_l)q_{12})(-\beta_1(T_f)\eta + \beta_2(T_f))e^{-\hat{\alpha}_G T_l} + \frac{m_1 + n_1}{2(\alpha_G + \tau)}(1 - e^{-2(\alpha_G + \tau)T_l}) \\ & + \frac{m_2 + n_2}{2\hat{\alpha}_G}(1 - e^{-2\hat{\alpha}_G T_l}) + \frac{m_3 + n_3}{\alpha_G + \tau + \hat{\alpha}_G}(1 - e^{-(\alpha_G + \tau + \hat{\alpha}_G)T_l}) \\ & + \frac{m_4 + n_4}{\alpha_G + \tau}(1 - e^{-(\alpha_G + \tau)T_l}) + \frac{m_5 + n_5}{\hat{\alpha}_G}(1 - e^{-\hat{\alpha}_G T_l}). \end{aligned}$$

To calculate the correlation coefficients of $R(t - T_l, t)$ and $R(t, t + T_f)$, we now calculate variances $\text{Var}\{R(t - T_l, t)\}$ and $\text{Var}\{R(t, t + T_f)\}$. Define

(A-92)

$$k_1 := \frac{a^2(\hat{\tau}_0^{1/2}\eta - \tau_0^{1/2})^2 + a^2N(\hat{\tau}_I^{1/2}\eta - \tau_I^{1/2})^2}{(\alpha_G + \tau)^2}, \quad k_2 := \frac{(-a\eta + b)^2(\hat{\tau}_0 + N\hat{\tau}_I)}{\hat{\alpha}_G^2},$$

(A-93)

$$k_3 := \frac{2a(-a\eta + b)(\hat{\tau}_0^{1/2}(\hat{\tau}_0^{1/2}\eta - \tau_0^{1/2}) + N\hat{\tau}_I^{1/2}(\hat{\tau}_I^{1/2}\eta - \tau_I^{1/2}))}{(\alpha_G + \tau)\hat{\alpha}_G},$$

(A-94)

$$c_{k1} := \frac{\hat{\tau}_0^{1/2}(-a\eta + b)}{\hat{\alpha}_G} + \frac{\sigma_G\Omega^{1/2}C_G\tau_0^{1/2} + \sigma_D(r + \alpha_G)}{(r + \alpha_D)(r + \alpha_G)} + \frac{a(\hat{\tau}_0^{1/2}\eta - \tau_0^{1/2})}{\alpha_G + \tau},$$

$$(A-95) \quad c_{k2} := N \left(\frac{-a(\tau_I^{1/2} - \hat{\tau}_I^{1/2}\eta)}{\alpha_G + \tau} + \frac{\hat{\tau}_I^{1/2}(-a\eta + b)}{\hat{\alpha}_G} + \frac{\sigma_G \Omega^{1/2} C_G \tau_I^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \right),$$

$$(A-96) \quad k_4 := -2a \frac{c_{k1} (\hat{\tau}_0^{1/2}\eta - \tau_0^{1/2}) + c_{k2} (\tau_I^{1/2}\eta - \tau_I^{1/2})}{\alpha_G + \tau},$$

$$(A-97) \quad k_5 := -\frac{2(-a\eta + b)(c_{k1} \hat{\tau}_0^{1/2} + c_{k2} \hat{\tau}_I^{1/2})}{\hat{\alpha}_G}, \quad k_6 := c_{k1}^2 + \frac{1}{N} c_{k2}^2.$$

Then direct calculations show that the variances $\text{Var}\{R(t-T_l, t)\}$ and $\text{Var}\{R(t, t+T_f)\}$ are as follows:

$$(A-98) \quad \begin{aligned} \text{Var}\{R(t-T_l, t)\} &= q_{11} \beta_1^2(T_l) + q_{22} \beta_2^2(T_l) - 2q_{12} \beta_1(T_l) \beta_2(T_l) \\ &+ \frac{k_1}{2(\alpha_G + \tau)} (1 - e^{-2(\alpha_G + \tau)T_l}) + \frac{k_2}{2\hat{\alpha}_G} (1 - e^{-2\hat{\alpha}_G T_l}) \\ &+ \frac{k_3}{\alpha_G + \tau + \hat{\alpha}_G} (1 - e^{-(\alpha_G + \tau + \hat{\alpha}_G)T_l}) + \frac{k_4}{\alpha_G + \tau} (1 - e^{-(\alpha_G + \tau)T_l}) + \frac{k_5}{\hat{\alpha}_G} (1 - e^{-\hat{\alpha}_G T_l}) + k_6 T_l, \end{aligned}$$

$$(A-99) \quad \begin{aligned} \text{Var}\{R(t, t+T_f)\} &= q_{11} \beta_1^2(T_f) + q_{22} \beta_2^2(T_f) - 2q_{12} \beta_1(T_f) \beta_2(T_f) \\ &+ \frac{k_1}{2(\alpha_G + \tau)} (1 - e^{-2(\alpha_G + \tau)T_f}) + \frac{k_2}{2\hat{\alpha}_G} (1 - e^{-2\hat{\alpha}_G T_f}) \\ &+ \frac{k_3}{\alpha_G + \tau + \hat{\alpha}_G} (1 - e^{-(\alpha_G + \tau + \hat{\alpha}_G)T_f}) + \frac{k_4}{\alpha_G + \tau} (1 - e^{-(\alpha_G + \tau)T_f}) + \frac{k_5}{\hat{\alpha}_G} (1 - e^{-\hat{\alpha}_G T_f}) + k_6 T_f. \end{aligned}$$

Then using equations (A-91), (A-98), and (A-99), we get the correlation coefficient of $R(t-T_l, t)$ and $R(t, t+T_f)$.

A.12. Proof of Proposition 4 and More Detailed Analysis on the Expected Returns

We now provide more detailed analysis on the expected returns for both relatively overconfident and absolutely overconfident cases.

We first look at the case when the economist and the traders agree on the total precision of information flow and other parameters of the model. Let

$$(A-100) \quad T_1 := \frac{1}{\alpha_G} \ln \left(\frac{(r + \alpha_G) \left(\tau - C_G \left(\tau_0 + N \hat{\tau}_I^{1/2} \tau_I^{1/2} \right) \right)}{(1 - C_G) \alpha_G (\alpha_G + \tau) + r \left(\alpha_G + \tau - C_G \left(\alpha_G + \tau_0 + N \hat{\tau}_I^{1/2} \tau_I^{1/2} \right) \right)} \right).$$

Direct computation shows that the coefficient of $\sum_{n=1}^N H_n(t)$ in proposition 4 is positive and monotonically increases in T . The coefficient of $H_0(t)$ monotonically increases in T for $T \geq T_1$.

In addition, as in equation (68), it can be shown that the coefficient $\beta_2(T)\hat{\tau}_I^{1/2} - \beta_1(T)\tau_I^{1/2} > 0$ of $\sum_{n=1}^N H_n(t)$ can be decomposed to two terms:

$$(A-101) \quad \frac{\sigma_G \Omega^{1/2}}{r + \alpha_D} (1 - C_G) \tau_I^{1/2} \frac{1 - e^{-\alpha_G T}}{\alpha_G} + \frac{\sigma_G \Omega^{1/2}}{(r + \alpha_D) \alpha_G (r + \alpha_G)} \left(\hat{\tau}_I^{1/2} - \tau_I^{1/2} \right) \cdot \left((1 - e^{-\alpha_G T}) \left(r + \alpha_G - \frac{C_G r \tau_0}{\alpha_G + \tau} \right) + \frac{C_G (\alpha_G + r + \tau) \alpha_G \tau_0}{(\alpha_G + \tau) \tau} (e^{-\alpha_G T} - e^{-(\alpha_G + \tau) T}) \right).$$

As before, the first term with $1 - C_G > 0$ results from the price dampening effect of the Keynesian beauty contest and the second term with $\hat{\tau}_I^{1/2} - \tau_I^{1/2} > 0$ results from the price dampening effect of beliefs aggregation.

Next, we look at the general case when investors and the economist may disagree about the total precision of the signals and parameter values of the model. The results when the economist and investors agree about the parameter values can be obtained by setting $\hat{\alpha}_G = \alpha_G$ and $\hat{\sigma}_G = \sigma_G$. We assume $\alpha_G + \tau > \hat{\alpha}_G$.⁸ Define

$$(A-102) \quad \nu_1 := \frac{\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2}}{\alpha_G - \hat{\alpha}_G + \tau}, \quad \nu_3 := \frac{C_G (r + \hat{\alpha}_G) (2\hat{\alpha}_G + \hat{\tau})^{1/2}}{(r + \alpha_G) (2\alpha_G + \tau)^{1/2}} \nu_1,$$

$$(A-103) \quad \nu_2 = \nu_1 + \frac{(r + \alpha_G) \hat{\Omega}^{1/2} (\hat{\sigma}_G - \nu_3 \sigma_G)}{C_G (r + \alpha_G + \tau) \Omega^{1/2} \sigma_G},$$

and

$$(A-104) \quad T_2 := \frac{1}{\alpha_G + \tau - \hat{\alpha}_G} \ln \left(\frac{C_G \sigma_G (r + \alpha_G + \tau) \Omega^{1/2} (H(t) - \nu_1 \hat{H}(t))}{(r + \alpha_G) \hat{\Omega}^{1/2} (\hat{\sigma}_G - \nu_3 \sigma_G) \hat{H}(t)} \right),$$

it can be shown that $T_2 > 0$ if and only if $H(t) > \nu_2 \hat{H}(t)$. We also have

$$(A-105) \quad \frac{dE_t\{R(t, t+T)\}}{dT} = \frac{e^{-(\alpha_G + \tau)T}}{(r + \alpha_D) (r + \alpha_G)} \left(-C_G \sigma_G (r + \alpha_G + \tau) \Omega^{1/2} (H(t) - \nu_1 \hat{H}(t)) + (r + \alpha_G) \hat{\Omega}^{1/2} (\hat{\sigma}_G - \nu_3 \sigma_G) \hat{H}(t) e^{(\alpha_G + \tau - \hat{\alpha}_G)T} \right).$$

The following proposition shows that there are only four possible patterns of the expected holding period return: only momentum, only mean-reversion, first mean-reversion and then momentum, first momentum and then mean-reversion.

⁸The case with $\alpha_G + \tau < \hat{\alpha}_G$ is similar.

PROPOSITION 5: *When investors and the economist may disagree about both the total precision of the signals and parameter values of the model, we have*

- 1) *if $H(t) \leq \nu_1 \hat{H}(t)$ and $\hat{H}(\hat{\sigma}_G - \nu_3 \sigma_G) \geq 0$, then $E_t\{R(t, t+T)\}$ monotonically increases in T ;*
- 2) *if $H(t) \leq \nu_1 \hat{H}(t)$ and $\hat{H}(\hat{\sigma}_G - \nu_3 \sigma_G) < 0$, then $E_t\{R(t, t+T)\}$ increases in T for $T < T_2$ and decreases in T for $T > T_2$;*
- 3) *if $H(t) > \nu_1 \hat{H}(t)$ and $\hat{H}(\hat{\sigma}_G - \nu_3 \sigma_G) \leq 0$, then $E_t\{R(t, t+T)\}$ monotonically decreases in T ;*
- 4) *if $H(t) > \nu_1 \hat{H}(t)$ and $\hat{H}(\hat{\sigma}_G - \nu_3 \sigma_G) > 0$, then $E_t\{R(t, t+T)\}$ decreases in T for $T < T_2$ and increases in T for $T > T_2$.*

Proposition 5 implies that the expected holding period returns $E_t\{R(t, t+T)\}$ can be monotonically increasing or decreasing over time T or it might be increasing first then decreasing over time T or it might be decreasing first then increasing over time T . Whether $E_t\{R(t, t+T)\}$ increases or decreases in time T depends on the relative magnitude of the current signals of $H(t)$ and $\hat{H}(t)$. It also depends on the disagreement on the mean-reverting rate and volatility of the dividend growth, α_G and σ_G , between the economist and investors. $E_t\{R(t, t+T)\}$ converges to a constant when $T \rightarrow \infty$. As illustrated in Proposition 5 and Figure 8, our model may generate short-run momentum and long-run reversal in the term structure of the returns as observed in the data when the economist disagrees with investors about the precisions, mean-reverting rate and the volatility of the growth rate of the dividend, i.e., $\hat{\tau} < \tau$, $\hat{\sigma}_G \neq \sigma_G$, and $\hat{\alpha}_G \neq \alpha_G$.

Assume investors are absolutely overconfident in the sense that $\hat{\tau} < \tau$, and assume the economist and investors agree that $\hat{\alpha}_G = \alpha_G$ and $\hat{\sigma}_G = \sigma_G$. For this case, it can be shown that

$$\nu_3 = \frac{C_G(2\alpha_G + \hat{\tau})^{1/2}}{(2\alpha_G + \tau)^{1/2}} \frac{\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2}}{\tau} < 1,$$

since $\hat{\Omega} > \Omega$, $C_G < 1$, and $\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2} < \tau$. This implies that we will obtain cases (1) and (4) in proposition 5 for positive signals $\hat{H}(t)$; specifically, we obtain (1) only momentum or (4) short-run reversal and long-run momentum as illustrated in figure 6.

From proposition 5, if $H(t)/\nu_1 \leq \hat{H}(t) < H(t)/\nu_2$, and

$$(A-106) \quad \sigma_G > \frac{(r + \alpha_G)(2\alpha_G + \tau)^{1/2}}{C_G(r + \hat{\alpha}_G)(2\hat{\alpha}_G + \hat{\tau})^{1/2}} \frac{\alpha_G - \hat{\alpha}_G + \tau}{\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_I^{1/2}} \hat{\sigma}_G,$$

where ν_1 and ν_2 are defined in (A-102) and (A-103). Equation (A-106) implies that we tend to have short-run momentum and long-run reversal if traders believe

that the growth rate is highly volatile (high values of σ_G) or highly persistent (low values of α_G).